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Regular Groups of Automorphisms of Cubic Graphs*

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INTRODUCTION

This paper is concerned with s-regular groups of automorphisms of connected cubic graphs. (See next section for definitions.) The study of the subject was started by Tutte [12] who proved that $s \leqslant 5$ in the case of finite cubic graphs (see also [15]). He also proved in that case that if the graph is 1-transitive then it must be s-regular for some s, $1 \leqslant s \leqslant 5$. These results remain valid for infinite cubic graphs with the exception of the infinite cubic tree.

Let G be a connected cubic graph and let $e = \{u, v\}$ be an edge of G. Let A be an s-regular subgroup of $\mathrm{Aut}(G)$. Then we associate to A the amalgam $(A(u), A(e))$, where $A(u)$ is the fixer of u in A and $A(e)$ is the stabilizer of $\{u, v\}$ in A. This amalgam is independent (up to isomorphism) of the chosen edge $\{u, v\}$. Moreover, if $s = 1, 3$, or 5 this amalgam is even independent of the choice of G and A; if $s = 2$ or 4 there are two possible amalgams in each case. Hence we obtain seven possible types of s-regular groups ($1 \leqslant s \leqslant 5$): $1'$, $2'$, $2''$, $3'$, $4'$, $4''$, and $5'$. The group of type s' is s-regular and has an involution which flips an edge. The groups of type s'' is s-regular and it has no involution flipping an edge.

Let Γ_3 be the cubic tree. We show that $\mathrm{Aut}(\Gamma_3)$ contains regular subgroups

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of all seven types, moreover regular subgroups of the same type are conjugate in $\mathrm{Aut}(\Gamma_3)$.

The possible inclusions among regular subgroups of $\mathrm{Aut}(\Gamma_3)$ are determined. For instance, an s-regular group with $s = 2$ or 3 is not contained in any t-regular subgroup with $t = 4$ or 5. Also a 1-regular subgroup is contained in precisely one subgroup of type $2'$, one subgroup of type $3'$, two subgroups of type $4'$, two subgroups of type $5'$ and it is not contained in any subgroup of type $2''$ or $4''$. Some of these assertions are valid in arbitrary cubic graphs. We determine the centralizers and normalizers in $\mathrm{Aut}(\Gamma_3)$ of s-regular subgroups ($1 \leqslant s \leqslant 5$). It is easy to see that a subgroup of type $1'$ of $\mathrm{Aut}(\Gamma_3)$ is isomorphic to the modular group $C_2 * C_3$ (the free product of a cyclic group of order 2 and a cyclic group of order 3).

It was shown by the first author in [5] that if A'_5 is a subgroup of type $5'$ of $\mathrm{Aut}(\Gamma_3)$ then the pair (Γ_3, A'_5) has a certain universal property. Namely, if (G, A) is another pair consisting of a connected cubic graph G and a subgroup A of type $5'$ of $\mathrm{Aut}(G)$ then there is a covering map $\Gamma_3 \to G$ which is compatible with the actions of A'_5 on Γ_3 and A on G. This gives rise to a homomorphism $f\colon A'_5 \to A$ which is onto and one can reconstruct the pair (G, A) from the pair (Γ_3, A'_5) and the knowledge of the subgroup $\ker f$ of A'_5.

Now, these results are proved in detail in this paper for all seven types of regular groups. The proofs are somewhat simpler then those used in [5].

It turns out that to all but a finite number of the normal subgroups N of the s-regular subgroup A of $\mathrm{Aut}(\Gamma_3)$ one can canonically associate a pair (G_N, A_N). This pair consists of a connected cubic graph G_N and an s-regular subgroup A_N of $\mathrm{Aut}(G_N)$ having the same type as A. Moreover, in this way we obtain all such pairs (G, B) (up to isomorphism). This correspondence $N \to (G_N, A_N)$ is either one–one or two–one, i.e., the same pair can be obtained from at most two normal subgroups. Hence the problem of classifying (G, B) of the given type reduces to the problem of classifying normal subgroups of A.

This latter problem is still far from its solution. For instance, if we consider the type $1'$ then A is the modular group and in spite of an enormous number of papers on normal subgroups of it (see the references in [9]) the final answer is still lacking. It is not even known what the simple quotients of the modular group are, i.e., which simple groups can be generated by one element of order 2 and one element of order 3.

These are the main ideas of this paper but an interested reader will find in it many other results which have not been mentioned above. As a sample, we now mention the following result: If G is a connected cubic graph and $\mathrm{Aut}(G)$ contains both a 2-regular and a 4-regular subgroup then G is a cubic tree.

1. Preliminaries

Our graphs have no loops nor multiple edges. For a graph G we write $G=(V, E)$ to indicate that V is the set of vertices and E the set of edges of G. If $a \in V$ we denote by $G(a)$ the set of neighbors of a in G.

A regular tree n will be denoted by Γ_n.

The graph *Path* (s), $s \geqslant 0$, is the graph (V, E) where $V=\{0, 1, \ldots, s\}$ and E consists of the pairs $\{i, i+1\}$, $0 \leqslant i \leqslant s-1$. The graph *Cir* (s), $s \geqslant 3$, is the graph obtained from *Path* $(s-1)$ by adding the edge $\{0, s-1\}$.

A *path of length s* (or, *an s-path*) in a graph G is a homomorphism f: *Path* $(s) \to G$. For such path we say that $f(0)$ is its *origin* and $f(s)$ its *end* and that *it joins* $f(0)$ to $f(s)$. An s-path is *closed* if $f(0)=f(s)$, otherwise it is *open*.

If f is an s-path in G then its *opposite* s-path f' is defined by $f'(i)=f(s-i)$, $0 \leqslant i \leqslant s$.

A graph homomorphism $f\colon G \to G'$ is *locally injective* (resp., *locally surjective*) if for every vertex a of G the map $f_a\colon G(a) \to G'(f(a))$, obtained form f by restricting its domain and codomain, is injective (resp., surjective). If f is both locally injective and locally surjective then we shall say that it is a *local isomorphism.*

An s-arc in a graph G is an s-path f which is locally injective, i.e., such that $f(i) \neq f(i+2)$ for $0 \leqslant i \leqslant s-2$. A 0-arc may be identified with its origin (=end). A 1-arc is also called an *oriented edge.* If f is an s-arc in G then the opposite s-path f' is also an s-arc.

A graph homomorphism $f\colon G \to G'$ is called a *covering* if both G and G' are connected and f is a local isomorphism. Every covering is surjective.

If S is an s-arc and $\alpha \in \operatorname{Aut}(G)$ then $\alpha \circ S$ is also an s-arc. In this way every α induces a permutation of the set of s-arcs in G.

A subgroup $A \leqslant \operatorname{Aut}(G)$ is *s-transitive* (resp., *s-regular*) if the action of A on the set of s-arcs of G is transitive (resp., regular, i.e., sharply transitive). A is *ω-transitive* if it is s-transitive for all integers $s \geqslant 0$. A graph G is *s-transitive* (resp., *s-regular*) if $\operatorname{Aut}(G)$ is s-transitive (resp., s-regular).

If A is s-transitive then it is also t-transitive for $0 \leqslant t \leqslant s$. It is clear that 0-transitive means vertex-transitive and that every 1-transitive group is also edge-transitive.

We will consider the category whose *objects* are ordered pairs (G, A) consisting of a graph G and a group A acting on it. The *morphisms* in this category are pairs (f, g) of maps

$$(f, g)\colon\ (G, A) \to (G', A'),$$

where $f\colon\ G \to G'$ is a graph homomorphism, $g\colon\ A \to A'$ a group homomorphism, and they are compatible in the sense that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & G \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ G' & \xrightarrow[g(\alpha)]{} & G' \end{array}$$

is commutative, i.e., $g(\alpha) \circ f = f \circ \alpha$, for all $\alpha \in A$.

A *covering morphism* is a morphism (f, g): $(G, A) \to (G', A')$ such that f is a graph covering and g is a surjective homomorphism.

We shall say that a graph covering f: $G \to G'$ is *compatible* with the action of a group A on G if $f(a) = f(b)$ implies that $f(\alpha \cdot a) = f(\alpha \cdot b)$ for all a, $b \in V$ and $\alpha \in A$. If this is so then we can define an action of A on G' as follows: if $x \in V'$ choose $a \in V$ such that $f(a) = x$ and define $\alpha \cdot x = f(\alpha \cdot a)$ for all $\alpha \in A$. We shall say that this action of A on G' is *induced* by the action of A on G.

A *shunt* in a graph G is an ordered pair (a, α) where a is a vertex of G and α an automorphism of G such that $\alpha(a)$ is adjacent to a and $\alpha^2(a) \neq a$. Every shunt (a, α) determines a doubly *infinite arc* S in G be defining $S(i) = \alpha^i(a)$, $i \in \mathbf{Z}$(the set of rational integers). The image of S in G will be called the *trajectory* of (a, α).

Let $A \leqslant \mathrm{Aut}(G)$. Then a shunt (a, α) in G is an *A-shunt* if $\alpha \in A$. The group $\mathrm{Aut}(G)$ acts on the set of all shunts in G by defining

$$\beta \cdot (a, \alpha) = (\beta(a), \beta\alpha\beta^{-1})$$

for $\beta \in \mathrm{Aut}(G)$ and any shunt (a, α). We say that two shunts (a, α) and (b, β) are *A-conjugate* (*or conjugate in A*) if there exists $\gamma \in A$ such that $\gamma \cdot (a, \alpha) = (b, \beta)$. If $A = \mathrm{Aut}(G)$ then we shall say *conjugate* instead of A-conjugate.

If (a, α) is a shunt so is (a, α^{-1}) and we say that they are *opposite* to each other. We say that two shunts (a, α) and (b, β) have *overlap s* $(s \geqslant 1)$ if $a = b$, $\alpha^k(a) = \beta^k(a)$ for $0 \leqslant k \leqslant s$ and $\alpha^{-1}(a) \neq \beta^{-1}(a)$, $\alpha^{s+1}(a) \neq \beta^{s+1}(a)$.

A semidirect product S of two groups A and B will be denoted by $A \rtimes B = S$; A is a normal subgroup of S and B acts on A. Our group-theoretical notation is standard; we just mention that S_n is the symmetric group of degree n, D_n the dihedral group of order $2n$, and C_n is the cyclic group of order n.

Let $G = (V, E)$ be a graph and $A \leqslant \mathrm{Aut}(G)$. *The fixer in A* of a vertex $v \in V$ is the subgroup $A(v)$ of A consisting of all $\alpha \in A$ such that $\alpha(v) = v$. If $v_1, ..., v_k$ are vertices of G we put

$$A(v_1, ..., v_k) = \bigcap_{i=1}^{k} A(v_i).$$

The *even subgroup* A^+ of A is the subgroup generated by all $A(v)$, $v \in V$.

PROPOSITION 1. *Let G be connected, A 1-transitive and let $\{x, y\} \in E$. Choose $\xi \in A$ such that $\xi(x) = y$. Then*

(i) *A^+ is generated by $A(x)$ and $A(y)$;*
(ii) *A is generated by $A(x)$ and ξ;*
(iii) *$(A:A^+) = 1$ or 2;*
(iv) *$(A:A^+) = 2$ iff G is bipartite.*

We leave the proof of this proposition to the reader. It is similar to the proof of Proposition 17.8 in [1, p. 117]. Part (ii) is identical to Lemma 2.3 of Miller [8].

2. VERTEX-FIXERS AND EDGE-STABILIZERS

Let $G = (V, E)$ be a connected cubic graph A an s-regular group of automorphisms of G, s being a positive integer. It is well known [13] that we must have $s \leqslant 5$ and that the girth of G is $\geqslant 2s - 2$. The s-regularity of A implies that $|A(x)| = 3 \cdot 2^{s-1}$ for every $x \in V$. Similarly if $e = \{x, y\} \in E$ then $|A(x, y)| = 2^{s-1}$. If also $\{y, z\} \in E$, $x \neq z$ and $s \geqslant 2$ then $|A(x, y, z)| = 2^{s-2}$, etc. The *stabilizer* of the edge $e = \{x, y\}$ in A is the subgroup $A(e)$ of A consisting of all $\alpha \in A$ such that $\alpha(e) = e$, i.e., either $\alpha(x) = x$ and $\alpha(y) = y$ or $\alpha(x) = y$ and $\alpha(y) = x$. Clearly $A(x, y)$ is a subgroup of $A(e)$ of index 2.

Our goal in this section is to describe the structure of the groups $A(x)$ and $A(e)$ where $e = \{x, y\} \in E$ in terms of certain canonical generators. Of course we have $A(x) \cap A(e) = A(x, y)$.

If $s = 1$ the problem is trivial. Then $A(x) \cong C_3$, $A(e) \cong C_2$ and $A(x, y)$ is the trivial group.

PROPOSITION 2. *Let $s = 2$ and let a, b, c, d be edges and x, y, z, u, v vertices as shown in Fig. 1. Let $\xi \in A$ be defined by $\xi(x) = u$, $\xi(y) = z$, $\xi(z) = y$. For each edge, say $a = \{x, y\}$ let $\tilde{a}$ be the non-trivial element of $A(x, y)$. Then we have*

(i) *$\tilde{a}$ is an involution and it moves every vertex at distance 1 from a;*
(ii) *if $\alpha \in A$ and $\alpha(a) = e$ then $\alpha\tilde{a}\alpha^{-1} = \tilde{e}$;*

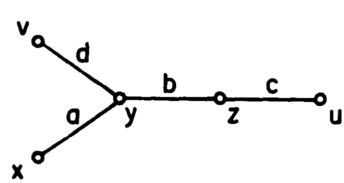

FIGURE 1

(iii) $(\tilde{a}\tilde{b})^3 = 1$, $\tilde{a}\tilde{b}\tilde{a} = \tilde{d}$;

(iv) $A(y) = \langle \tilde{a}, \tilde{b} \rangle \cong D_3$;

(v) *either* $\xi^2 = 1$ *and* $A(b) = \langle \tilde{b}, \xi \rangle \cong C_2 \times C_2$, *or* $\xi^2 = \tilde{b}$ *and* $A(b) = \langle \xi \rangle \cong C_4$.

Proof. (i) This follows from $|A(x, y)| = 2$ and the 2-regularity of A.

(ii) Follows from the fact that $\alpha\tilde{a}\alpha^{-1}$ fixes the vertices $\alpha(x)$ and $\alpha(y)$ of e.

(iii) Since $\tilde{a}(b) = d$ by (i), we have $\tilde{a}\tilde{b}\tilde{a} = \tilde{d}$ by (ii). Similarly, $\tilde{b}\tilde{a}\tilde{b} = \tilde{d}$ and hence $(\tilde{a}\tilde{b})^3 = (\tilde{a}\tilde{b}\tilde{a})(\tilde{b}\tilde{a}\tilde{b}) = (\tilde{d})^2 = 1$.

(iv) Since $\tilde{a} \neq \tilde{b}$ and both belong to $A(y)$ which has order 6, it follows that $A(y) = \langle \tilde{a}, \tilde{b} \rangle \cong D_3$.

(v) Since $\xi^2 \in A(y, z) = \langle \tilde{b} \rangle$ we have either $\xi^2 = 1$ or $\xi^2 = \tilde{b}$. The rest is clear.

PROPOSITION 3. *Let $s = 3$ and let a, b, c, d, e be vertices of G as in Fig. 2. Let $\xi \in A$ be defined by $\xi(a) = d$, $\xi(b) = c$, $\xi(c) = b$, $\xi(d) = a$. Then we have*

(i) *for each vertex, say $b \in V$, there is a unique non-trivial element $\tilde{b} \in A$ which fixes b and all its neighbors*;

(ii) *$\tilde{b}$ is an involution and it moves every vertex at distance 2 from b*;

(iii) *if $\alpha \in A$ and $\alpha(b) = x$ then $\alpha\tilde{b}\alpha^{-1} = \tilde{x}$*;

(iv) *$\tilde{b}$ belongs to the center of $A(b)$*;

(v) $A(a, b) = \langle \tilde{a}, \tilde{b} \rangle \cong C_2 \times C_2$, $A(b) = \langle \tilde{a}, \tilde{b}, \tilde{c} \rangle = \langle \tilde{a}, \tilde{c} \rangle \times \langle \tilde{b} \rangle$ *and* $\langle \tilde{a}, \tilde{c} \rangle \cong D_3$;

(vi) $\xi^2 = 1$, $\xi\tilde{b}\xi = \tilde{c}$ *and if* $u = \{b, c\}$ *then* $A(u) = \langle \tilde{b}, \xi \rangle \cong D_4$.

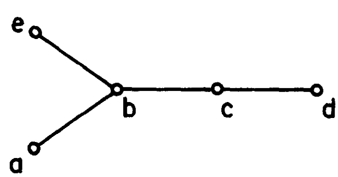

FIGURE 2

Proof. (i) and (ii) follow from $A(a, b, c) = A(a, b, c, e)$, $|A(a, b, c)| = 2$ and the 3-regularity of A. (iii) follows from the fact that $\alpha\tilde{b}\alpha^{-1}$ fixes x and all its neighbors. (iv) follows from (iii) by taking $\alpha \in A(b)$.

(v) Since $\tilde{a}(b) = b$ we have $\tilde{a}\tilde{b}\tilde{a} = \tilde{b}$ by (iii), and so $A(a, b) = \langle \tilde{a}, \tilde{b} \rangle \cong C_2 \times C_2$. Since $\tilde{c}(a) = e$ we have $\tilde{c} \notin A(a, b)$ and hence $A(b) = \langle \tilde{a}, \tilde{b}, \tilde{c} \rangle$. Since $\tilde{a}(c) = e$ we have $\tilde{a}\tilde{c}\tilde{a} = \tilde{e}$ and similarly $\tilde{c}\tilde{a}\tilde{c} = \tilde{e}$. Thus $(\tilde{a}\tilde{c})^3 = (\tilde{a}\tilde{c}\tilde{a}) \cdot (\tilde{c}\tilde{a}\tilde{c}) = (\tilde{e})^2 = 1$. Hence $\langle \tilde{a}, \tilde{c} \rangle \cong D_3$ and $A(b) = \langle \tilde{a}, \tilde{c} \rangle \times \langle \tilde{b} \rangle$ because $\tilde{b}$ commutes with both $\tilde{a}$ and $\tilde{c}$.

(vi) We have $\xi^2 = 1$ because it fixes a, b, c, d. Since $\xi(b) = c$, we have $\xi\tilde{b}\xi = \tilde{c}$ by (iii). The rest is clear.

PROPOSITION 4. *Let $s = 4$ and let a, b, c, d, e, f be the edges and $v_0, \ldots, v_7$ the vertices as indicated on Fig. 3. Let $\xi \in A$ be defined by $\xi(v_i) = v_{5-i}$ for $0 \leqslant i \leqslant 4$. Then we have*

(i) *for each edge, say b, there exists a unique non-trivial element $\tilde{b} \in A$ which fixes all vertices at distance $\leqslant 1$ from b;*

(iii) *$\tilde{b}$ is an involution and it moves every vertex at distance 2 from b;*

(iii) *if $\alpha \in A$ and $\alpha(b) = x$ then $\alpha\tilde{b}\alpha^{-1} = \tilde{x}$;*

(iv) *$\tilde{b}$ belongs to the center of $A(b)$;*

(v) *$\tilde{a}\tilde{b} = \tilde{b}\tilde{a}$, $(\tilde{a}\tilde{c})^2 = \tilde{b}$, $(\tilde{a}\tilde{d})^3 = 1$;*

(vi) *$A(v_0, v_1, v_2) = \langle \tilde{a}, \tilde{b} \rangle \cong C_2 \times C_2$, $A(v_1, v_2) = \langle \tilde{a}, \tilde{b}, \tilde{c} \rangle = \langle \tilde{a}, \tilde{c} \rangle \cong D_4$, $A(v_2) = \langle \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \rangle \cong S_4$;*

(vii) *either $\xi^2 = 1$ and $A(c) = \langle \tilde{b}, \xi \rangle \cong D_8$ or $\xi^2 = \tilde{c}$ and $A(c) = \langle \tilde{b}, \xi \rangle \cong \tilde{D}_8$ is a quasidihedral group, see* [7, *p.* 90, *Satz* 14.9]. *In the latter case $\xi\tilde{b}$ has order* 8 *and $\tilde{b}(\xi\tilde{b})\tilde{b} = (\xi\tilde{b})^3 = \xi\tilde{d}$.*

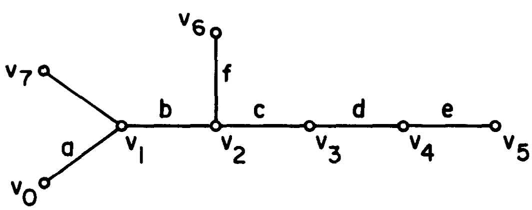

FIGURE 3

Proof. The proofs of (i)–(iv) we leave to the reader.

(v) Since $\tilde{a}(b) = b$ we have $\tilde{a}\tilde{b}\tilde{a} = \tilde{b}$ by (iii), and so $\tilde{a}\tilde{b} = \tilde{b}\tilde{a}$. Similarly $\tilde{a}\tilde{c}\tilde{a} = \tilde{f}$ and since $\tilde{b}$, $\tilde{c}$, $\tilde{f}$ are the non-trivial elements of $A(v_1, v_2, v_3) \cong C_2 \times C_2$, we have $(\tilde{a}\tilde{c})^2 = \tilde{f}\tilde{c} = \tilde{b}$.

Since $(\tilde{d}\tilde{a})(v_2) = v_2$, and $(\tilde{d}\tilde{a})(v_3) = \tilde{d}(v_6) = v_1$, we must have $(\tilde{d}\tilde{a})(v_4) = v_0$ or v_7. In both cases this vertex is fixed by $\tilde{a}$, i.e., $\tilde{a}\tilde{d}\tilde{a}(v_4) = \tilde{d}\tilde{a}(v_4) = \tilde{d}\tilde{a}\tilde{d}(v_4)$ and so $(\tilde{a}\tilde{d})^3 \in A(v_2, v_4)$. Similarly, $(\tilde{a}\tilde{d})^3 \in A(v_0, v_2)$ and so $(\tilde{a}\tilde{d})^3 = 1$ by 4-regularity of A.

(vi) Here we shall only prove the last assertion. Since $\tilde{d}$ moves v_1, we have $\tilde{d} \notin A(v_1, v_2) = \langle \tilde{a}, \tilde{b}, \tilde{c} \rangle$ and so $A(v_2) = \langle \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \rangle$. (v) implies that $\langle \tilde{a}, \tilde{d} \rangle \cong D_3$. Since $\langle \tilde{b}, \tilde{c} \rangle = A(v_1, v_2, v_3) \lhd A(v_2)$ it follows that $A(v_2) \cong S_4$ [7, p. 201, Hilfssatz 8.17].

(vii) Since ξ^2 fixes v_i for $1 \leqslant i \leqslant 4$, we have $\xi^2 \in \langle \tilde{c} \rangle$.

If $\xi^2 = 1$ then $\xi(v_5) = v_0$, $(\xi\tilde{b})^2 = \xi\tilde{b}\xi\tilde{b} = \tilde{d}\tilde{b}$, $(\xi\tilde{b})^4 = (\tilde{d}\tilde{b})^2 = \tilde{c}$ and hence $\xi\tilde{b}$ has order 8. Then $A(c) = \langle \tilde{b}, \xi \rangle \cong D_8$.

Now let $\xi^2 = \tilde{c}$. Then $(\xi\tilde{b})^2 = \xi\tilde{b}\xi\tilde{b} = \xi\tilde{b}\xi^{-1}\,\tilde{c}\tilde{b} = \tilde{d}\tilde{c}\tilde{b} = \tilde{b}\tilde{d}$ and again $\xi\tilde{b}$ has order 8. We find that

$$\tilde{b}(\xi\tilde{b})\tilde{b} = \tilde{b}\xi = (\xi\tilde{d}\xi^{-1})\xi = \xi\tilde{d},$$
$$(\xi\tilde{b})^3 = \xi\tilde{b}(\xi\tilde{b})^2 = \xi\tilde{b}\cdot\tilde{b}\tilde{d} = \xi\tilde{d}.$$

Hence $A(c) = \langle \tilde{b}, \xi\rangle \cong \tilde{D}_8$.

PROPOSITION 5. *Let $s = 5$ and let $a, b, \ldots$ be the vertices as indicated in Fig. 4. Let $\xi \in A$ be defined by $\xi(a) = f$, $\xi(b) = e$, $\xi(c) = d$, $\xi(d) = c$, $\xi(e) = b$, $\xi(f) = a$.*

Then we have

(i) *for each vertex, say c, there is a unique non-trivial element $\tilde{c} \in A$ which fixes all vertices at distance $\leqslant 2$ from c;*

(ii) *$\tilde{c}$ is an involution and it moves every vertex at distance 3 from c;*

(iii) *if $\alpha \in A$ and $\alpha(c) = x$ then $\alpha\tilde{c}\alpha^{-1} = \tilde{x}$;*

(iv) *$\tilde{c}$ belongs to the center of $A(c)$;*

(v) $\tilde{a}\tilde{b} = \tilde{b}\tilde{a}$, $\tilde{a}\tilde{c} = \tilde{c}\tilde{a}$, $(\tilde{a}\tilde{d})^2 = \tilde{b}\tilde{c}$, $(\tilde{a}\tilde{e})^3 = 1$;

(vi) $A(a, b, c, d) = \langle \tilde{b}, \tilde{c}\rangle \cong C_2 \times C_2$, $A(b, c, d) = \langle \tilde{b}, \tilde{c}, \tilde{d}\rangle \cong C_2 \times C_2 \times C_2$, $A(b, c) = \langle \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\rangle = \langle \tilde{a}, \tilde{d}\rangle \times \langle \tilde{b}\rangle = \langle \tilde{a}, \tilde{d}\rangle \times \langle \tilde{c}\rangle$, $\langle \tilde{a}, \tilde{d}\rangle \cong D_4$, $A(c) = \langle \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}\rangle = \langle \tilde{a}, \tilde{b}\tilde{c}, \tilde{c}\tilde{d}, \tilde{e}\rangle \times \langle \tilde{c}\rangle$, $\langle \tilde{a}, \tilde{b}\tilde{c}, \tilde{c}\tilde{d}, \tilde{e}\rangle \cong S_4$;

(vii) *$\xi^2 = 1$ and if $u = \{c, d\}$ then $A(u) = A(c, d) \rtimes \langle \xi\rangle$, where ξ acts on $A(c, d) = \langle \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}\rangle$ by*

$$\xi\tilde{b}\xi = \tilde{e}, \qquad \xi\tilde{c}\xi = \tilde{d}, \qquad \xi\tilde{d}\xi = \tilde{c}, \qquad \xi\tilde{e}\xi = \tilde{b}.$$

FIGURE 4

Proof. The claims (i)–(vi) are proved in [5, Lemma 1–3 and Theorem 1]. Since ξ^2 fixes a, b, c, d, e, f we have $\xi^2 = 1$ and (vii) follows.

It is easy to check that if $\xi^2 \neq 1$ ($s = 2$ or 4) then there is no involution $\alpha \in A$ which flips an edge. In that case we say that A is of *type s''*. Otherwise it is of *type s'*. Thus we have seven possible types $1'$, $2'$, $2''$, $3'$, $4'$, $4''$, $5'$.

3. SHUNTS

Let G and A be as before, $1 \leqslant s \leqslant 5$.

LEMMA 1. *There are precisely 2 orbits for the action of A on $(s+1)$-arcs. If S_1 and S_2 are two distinct $(s+1)$-arcs and $S_1(i) = S_2(i)$ for $0 \leqslant i \leqslant s$ then they are not in the same orbit.*

Proof. Let S_1 and S_2 be as in the lemma. If S is any $(s+1)$-arc then there exists $\alpha \in A$ such that $\alpha(S(i)) = S_1(i) = S_2(i)$ for $0 \leqslant i \leqslant s$. Hence $\alpha \circ S = S_1$ or S_2 and there are at most 2 orbits. But S_1 and S_2 must be in different orbits because A is s-regular.

To each $(s+1)$-arc S we shall associate an A-shunt $\mathrm{sh}(S) = (a, \alpha)$ by taking $a = S(0)$ and defining α to be the unique element of A such that $\alpha(S(i)) = S(i+1)$ for $0 \leqslant i \leqslant s$.

LEMMA 2. *There are precisely 2 A-conjugacy classes of A-shunts. Two A-shunts with overlap s are not conjugate in A.*

Proof. The map sh from $(s+1)$-arcs to A-shunts is bijective. It is also A-equivariant, i.e., we have

$$\alpha \cdot \mathrm{sh}(S) = \mathrm{sh}(\alpha \circ S)$$

for all $\alpha \in A$ and all $(s+1)$-arcs S.

Hence both assertions of Lemma 2 follow from the corresponding assertions of Lemma 1.

LEMMA 3. *If an A-shunt (a, α) is A-conjugate to its opposite (a, α^{-1}) then the same is true for all A-shunts.*

Proof. Assume that there is an A-shunt (b, β) which is not A-conjugate to (b, β^{-1}). Either (b, β) or (b, β^{-1}) is then A-conjugate to (a, α), say $\gamma \cdot (b, \beta) = (a, \alpha)$. By hypothesis there exists $\delta \in A$ such that $\delta \cdot (a, \alpha) = (a, \alpha^{-1})$. Then

$$\gamma^{-1} \delta \gamma \cdot (b, \beta) = \gamma^{-1} \delta(a, \alpha) = \gamma^{-1}(a, \alpha^{-1}) = (b, \beta^{-1})$$

gives a contradiction.

We shall say that A is of the *first* (*resp., second*) *kind* if every A-shunt is (resp. is not) A-conjugate to its opposite. Tutte [14] has shown that 1-regular groups are of the second kind. This is included in the following proposition.

PROPOSITION 6. *Groups of type $2'$, $3'$, $4'$, $5'$ are of the first kind and groups of type $1'$, $2''$, $4''$ are of the second kind.*

Proof. If A is of type $1'$ let $\{u, v\}$ be an edge of G, $G(u) = \{a, b, v\}$, and $G(v) = \{u, x, y\}$. There is a $\rho \in A$ such that $\rho(v) = a$, $\rho(a) = b$, $\rho(b) = v$ and a $\xi \in A$ such that $\xi(u) = v$, $\xi(v) = u$. Then $\xi^2 = \rho^3 = 1$ and $(u, \xi\rho)$ is a shunt because $\xi\rho(u) = \xi(u) = v$ and $\xi\rho(v) = \xi(a) = x$ or y. Its opposite shunt is $(u, \rho^{-1}\xi)$. Since $\rho \cdot (u, \rho^{-1}\xi) = (u, \xi\rho^{-1})$ and $(u, \xi\rho)$ have overlap 1, Lemma 2 implies that they are not A-conjugate and so A is of the second kind.

Now let A be of type $2'$, $3'$, $4'$, or $5'$. Let u be a vertex of G and $\alpha \in A(u)$

an involution. Choose a vertex a so that $\alpha(a) \neq a$ and the distance $\delta(u, a)$ is minimal. Some neighbor, say b, of a is fixed by α. There is an involution $\beta \in A$ which flips the edge $\{a, b\}$. Then $(b, \alpha\beta)$ is a shunt because $\alpha\beta(a) = \alpha(b) = b$ and $\alpha\beta(b) = \alpha(a) \neq a$. Since $\alpha \cdot (b, \alpha\beta) = (b, \beta\alpha)$ we see that $(b, \alpha\beta)$ is A-conjugate to its opposite shunt $(b, \beta\alpha)$. Thus A is of the first kind.

Next, let A be of type $2''$. We shall use notation from Fig. 1 and the element ξ defined in Proposition 2, $\xi^2 = \tilde{b}$. Then $(x, \xi\tilde{d})$ and $(x, \xi^{-1}\tilde{d})$ are two shunts with overlap 2. Indeed, we have

$$\xi\tilde{d}(x) = \xi(z) = y, \qquad \xi\tilde{d}(y) = \xi(y) = z;$$
$$\xi^{-1}\tilde{d}(x) = \xi^{-1}(z) = y, \qquad \xi^{-1}\tilde{d}(y) = \xi^{-1}(y) = z.$$

Since $\tilde{d} \cdot (x, \tilde{d}\xi^{-1}) = (z, \xi^{-1}\tilde{d})$ and $(\tilde{d}\xi)^2 \cdot (z, \xi^{-1}\tilde{d}) = (x, \xi^{-1}\tilde{d})$, Lemma 2 implies that $(x, \xi\tilde{d})$ and its opposite $(x, \tilde{d}\xi^{-1})$ are not A-conjugate. Thus A is of the second kind.

Finally, let A be of type $4''$. Now we shall use Fig. 3 and the element ξ defined in Proposition 4, $\xi^2 = \tilde{c}$. The shunts $(v_0, \xi\tilde{a}\tilde{d}\tilde{a})$ and $(v_0, \xi^{-1}\tilde{a}\tilde{d}\tilde{a})$ have overlap 4 because

$$\xi\tilde{a}\tilde{d}\tilde{a}(v_0) = \xi\tilde{a}\tilde{d}(v_0) = \xi(v_4) = v_1,$$
$$\xi\tilde{a}\tilde{d}\tilde{a}(v_1) = \xi\tilde{a}\tilde{d}(v_1) = \xi\tilde{a}(v_6) = \xi(v_3) = v_2,$$
$$\xi\tilde{a}\tilde{d}\tilde{a}(v_2) = \xi(v_2) = v_3,$$
$$\xi\tilde{a}\tilde{d}\tilde{a}(v_3) = \xi\tilde{a}\tilde{d}(v_6) = \xi\tilde{a}(v_1) = \xi(v_1) = v_4,$$

and similarly $\xi^{-1}\tilde{a}\tilde{d}\tilde{a}(v_i) = v_{i+1}$ for $0 \leqslant i \leqslant 3$. Only the step $\tilde{a}\tilde{d}(v_0) = v_4$ needs justification. Since $\tilde{a}\tilde{d}(v_1) = \tilde{a}(v_6) = v_3$ it follows that $\tilde{a}\tilde{d}(v_0)$ is a neighbor of v_3. If we put $x = \tilde{a}\tilde{d}(a)$ then $\tilde{x} = (\tilde{a}\tilde{d})\,\tilde{a}(\tilde{d}\tilde{a})$, i.e., $\tilde{x} = \tilde{d}$. Since v_3 is a vertex of x this forces $x = d$ and hence $\tilde{a}\tilde{d}(v_0) = v_4$.

Hence $\tilde{a}\tilde{d}\tilde{a} \cdot (v_0, \tilde{a}\tilde{d}\tilde{a}\xi) = (v_4, \xi\tilde{a}\tilde{d}\tilde{a})$ and $(\xi\tilde{a}\tilde{d}\tilde{a})^4 \cdot (v_0, \xi\tilde{a}\tilde{d}\tilde{a}) = (v_4, \xi\tilde{a}\tilde{d}\tilde{a})$. Now the argument is the same as for type $2''$.

The proof is completed.

PROPOSITION 7. *Let A be an s-regular subgroup of* Aut(G), $1 \leqslant s \leqslant 5$. *Then A is generated by any two A-shunts with overlap s.*

This is an immediate consequence of [1, Theorem 17.5, p. 115].

PROPOSITION 8. *Let (a, α) be a shunt in Γ_3. Then*

(i) *there exists an involution $\beta \in$* Aut(Γ_3) *such that $A = \langle \alpha, \beta \rangle$ is ω-transitive;*

(ii) *there exists a shunt (a, γ) having overlap* 1 *with (a, α) and such that $B = \langle \alpha, \gamma \rangle$ is ω-transitive.*

Proof. (i) Let $a_i = \alpha^i(a)$ for $i \in \mathbf{Z}$. Choose an involution β such that $\beta(a_i) = a_i$ for $0 \leqslant i \leqslant 6$, $\beta(a_{-1}) \neq a_{-1}$, and $\beta(a_7) \neq a_7$. Then the shunts (a, α) and $\beta \cdot (a, \alpha) = (a, \beta\alpha\beta)$ have overlap 6. By [1, Theorem 17.5] the group $\langle \alpha, \beta\alpha\beta \rangle$ is 6-transitive. Since it cannot be 6-regular it follows [12, p. 63] that it is 7-transitive. Continuing in this way we see that $\langle \alpha, \beta\alpha\beta \rangle$ (and also A) is ω-transitive.

(ii) Since Γ_3 is ω-transitive there exists $\gamma \in \mathrm{Aut}(\Gamma_3)$ such that $\gamma(\alpha^{-i}(a)) = \alpha^{i+1}(a)$ for $0 \leqslant i \leqslant 6$, but $\gamma(\alpha^{-i}(a)) \neq \alpha^{i+1}(a)$ for $i = -1$ or 7. Then (a, α) and (a, γ) have overlap 1. On the other hand, $(\alpha(a), \alpha)$ and $(\alpha(a), \gamma\alpha^{-1}\gamma^{-1})$ have overlap 6. Hence (as above) $\langle \alpha, \gamma\alpha^{-1}\gamma^{-1} \rangle$ (and also B) is ω-transitive.

PROPOSITION 9. *Let A be an s-regular subgroup of* $\mathrm{Aut}(G)$, $2 \leqslant s \leqslant 5$. *Assume that B is a proper normal subgroup of A and that there is a shunt (a, β) with $\beta \in B$. Then $(A{:}\, B) = 2$ and B is $(s-1)$-regular.*

Proof. Since A is s-regular there exists a unique element $\alpha \in A$ which fixes $\beta^i(a)$ for $0 \leqslant i \leqslant s-1$ and $\alpha \neq 1$. Since α^2 fixes also $\beta^s(a)$ we have $\alpha^2 = 1$. Since $\alpha \neq 1$ the s-regularity of A implies that α moves $\beta^{-1}(a)$ and $\beta^s(a)$. Therefore the shunt $\alpha \cdot (a, \beta) = (a, \alpha\beta\alpha)$ has overlap $s-1$ with (a, β). Since $\beta \in B$, $\alpha \in A$ and $B \lhd A$ it follows that $(a, \alpha\beta\alpha)$ is also a B-shunt. Now [1, Theorem 17.5, p. 115] implies that B is $(s-1)$-transitive. Since $B \neq A$, B is $(s-1)$-regular and so $(A{:}\, B) = 2$.

4. SOME IMPORTANT AMALGAMS

An *amalgam* (more precisely, *group amalgam*) is an ordered pair (X, Y) where X, Y are groups such that $X \cap Y$ is a subgroup in both X and Y and the group structures in $X \cap Y$ induced by X and Y coincide. If (X, Y) and (X', Y') are amalgams then a *homomorphism* f: $(X, Y) \to (X, Y)$ is a map f: $X \cup Y \to X' \cup Y'$ such that $f^{-1}(X') = X$, $f^{-1}(Y') = Y$ and the restrictions f_X: $X \to X'$ and f_Y: $Y \to Y'$ are group homomorphisms. An injective (resp. bijective) homomorphism $(X, Y) \to (X', Y')$ is called an *embedding* (resp. *isomorphism*), etc.

If (X, Y) and (X', Y') are two amalgams such that $X \subset X'$, $Y \subset Y'$ and the inclusion map $X \cup Y \to X' \cup Y'$ is an embedding of amalgams, then we say that (X, Y) is a *subamalgam* of (X', Y'). Note that we then have $X' \cap Y = X \cap Y' = X' \cap Y'$.

Two subamalgams (X', Y') and (X'', Y'') of (X, Y) are said to be *conjugated* if there exists an $a \in X \cap Y$ such that $aX'a^{-1} = X''$ and $aY'a^{-1} = Y''$.

Now we describe several important amalgams.

AMALGAM 1′. (X_1, Y_1), where $X_1 = \langle a; a^3 = 1\rangle$, $Y_1 = \langle y; y^2 = 1\rangle$, $X_1 \cap Y_1 = \{1\}$.

AMALGAM 2′. (X_2, Y_2'), where $X_2 = \langle a, b; a^2 = b^2 = (ab)^3 = 1\rangle \cong D_3$, $Y_2' = \langle b, y; b^2 = y^2 = (yb)^2 = 1\rangle \cong C_2 \times C_2$ and $X_2 \cap Y_2' = \langle b\rangle \cong C_2$.

AMALGAM 2″. (X_2, Y_2''), where $X = \langle a, b; a^2 = b^2 = (ab)^3 = 1\rangle \cong D_3$, $Y'' = \langle y; y^4 = 1\rangle$, $y^2 = b$, $X_2 \cap Y_2'' = \langle b\rangle \cong C_2$.

AMALGAM 3′. (X_3, Y_3), where

$$X_3 = \langle a, b, c; a^2 = b^2 = c^2 = (ab)^2 = (bc)^2 = (ac)^3 = 1\rangle \cong D_6,$$
$$Y_3 = \langle b, c, y; b^2 = c^2 = y^2 = (bc)^2 = 1,\ yby = c\rangle \cong D_4,$$

and

$$X_3 \cap Y_3 = \langle b, c\rangle \cong C_2 \times C_2.$$

AMALGAM 4′. (X_4, Y_4'), where X_4 is generated by the elements a, b, c, d with defining relations

$$a^2 = b^2 = c^2 = d^2 = (ab)^2 = (bc)^2 = (cd)^2 = 1,$$
$$(ac)^2 = b, \qquad (bd)^2 = c, \qquad (ad)^3 = 1,$$

and Y_4' is generated by the elements b, c, d, y with defining relations

$$b^2 = c^2 = d^2 = y^2 = (bc)^2 = (cd)^2 = (yc)^2 = 1,$$
$$(bd)^2 = c, \qquad yby = d.$$

Then $X_4 \cong S_4$, $Y_4' \cong D_8$ and $X_4 \cap Y_4' = \langle b, c, d\rangle \cong D_4$, i.e., $X_4 \cap Y_4'$ is a Sylow 2-subgroup of X_4.

AMALGAM 4″. (X_4, Y_4''), where X_4 is the same as in the previous case and Y_4'' is generated by b, c, d, y with defining relations

$$b^2 = c^2 = d^2 = (bc)^2 = (cd)^2 = 1, \qquad (bd)^2 = c,$$
$$y^2 = c, \qquad yby^{-1} = d.$$

Then $Y_4'' \cong \tilde{D}_8$ (quasidihedral group of order 16) and $X_4 \cap Y_4'' = \langle b, c, d\rangle \cong D_4$.

AMALGAM 5′. (X_5, Y_5), where X_5 is generated by the elements a, b, c, d, e with defining relations

$$\begin{gathered} a^2 = b^2 = c^2 = d^2 = e^2 = (ab)^2 = (bc)^2 = (cd)^2 \\ = (de)^2 = (ac)^2 = (bd)^2 = (ce)^2 = 1, \\ (ad)^2 = bc, \qquad (be)^2 = cd, \qquad (ae)^3 = 1, \end{gathered}$$

and Y_5 is generated by the elements b, c, d, e, y with defining relations

$$\begin{gathered} b^2 = c^2 = d^2 = e^2 = y^2 = (bc)^2 = (cd)^2 = (de)^2 = (bd)^2 = (ce)^2 = 1, \\ (be)^2 = cd, \qquad yby = e, \qquad ycy = d. \end{gathered}$$

In this case

$$\begin{gathered} X_5 = \langle a, bc, cd, e\rangle \times \langle c\rangle, \\ \langle a, bc, cd, e\rangle \cong S_4, \qquad \langle c\rangle \cong C_2, \end{gathered}$$

and

$$Y_5 = \langle b, c, d, e\rangle \rtimes \langle y\rangle.$$

We have $X_5 \cap Y_5 = \langle b, c, d, e\rangle = \langle b, e\rangle \times \langle c\rangle = \langle b, e\rangle \times \langle d\rangle \cong D_4 \times C_2$.

Let A be an s-regular subgroup of $\mathrm{Aut}(G)$ where G is a connected cubic graph ($1 \leqslant s \leqslant 5$). To each 1-arc S of G, say $S(0) = u$, $S(1) = v$, $t = \{u, v\}$ we associate the amalgam $(A(u), A(t))$.

PROPOSITION 10. *The amalgam $(A(u), A(t))$ is isomorphic to (X_s, Y_s), where if $s = 2$ or 4 then $Y_s = Y_s'$ if A is of the first kind and $Y_s = Y_s''$ if A is of the second kind.*

Proof. This is immediate from the definition of the amalgams (X_s, Y_s) and from Propositions 2–5.

It is easy to check that the following are embeddings among the amalgams defined in this section:

$$\begin{aligned} f_{12}&: (X_1, Y_1) \to (X_2, Y_2'); \\ &\quad f_{12}(a) = ab, f_{12}(y) = y; \\ f_{23}'&: (X_2, Y_2') \to (X_3, Y_3), \\ &\quad f_{23}'(a) = ab, f_{23}'(b) = bc, f_{23}'(y) = y; \\ f_{23}''&: (X_2, Y_2'') \to (X_3, Y_3), \\ &\quad f_{23}''(a) = ab, f_{23}''(b) = bc, f_{23}''(y) = yb; \end{aligned}$$

$f_{14}\colon (X_1, Y_1) \to (X_4, Y_4')$
$$f_{14}(a) = ad, f_{14}(y) = y;$$

$f_{45}'\colon (X_4, Y_4') \to (X_5, Y_5)$,
$$f_{45}'(a) = ab, f_{45}'(b) = bc, f_{45}'(c) = cd, f_{45}'(d) = de, f_{45}'(y) = y;$$

$f_{45}''\colon (X_4, Y_4'') \to (X_5, Y_5)$,
$$f_{45}''(a) = ab, f_{45}''(b) = bc, f_{45}''(c) = cd, f_{45}''(d) = de, f_{45}''(y) = yc.$$

5. Regular Actions on Γ_3

Let (X_s, Y_s) be the amalgams constructed in the previous section where $Y_s = Y_s'$ or Y_s'' if $s = 2$ or 4. Put $H_s = X_s \cap Y_s$ and

$$\begin{aligned} A_s' &= X_s *_{H_s} Y_s \qquad (s = 1, 3, 5), \\ A_s' &= X_s *_{H_s} Y_s' \qquad (s = 2, 4), \\ A_s'' &= X_s *_{H_s} Y_s'' \qquad (s = 2, 4), \end{aligned}$$

where * denotes the free product with amalgamation.

We shall write A_s for A_s' or A_s''.

Let G_s be the graph whose vertex set is the disjoint union of A_s/X_s and A_s/Y_s and whose edge set consists of all $\{uX_s, uY_s\}$ for $u \in A_s$. Vertices of type uX_s have valence 3 and those of type uY_s have valence 2. Then G_s is a tree, see [10, Theorem 7, pp. 1–50 and 51]. (We shall use freely these widely circulated notes of Serre.) The group A_s acts on G_s by left multiplication. The fixer in A_s of the vertex X_s of G_s is the subgroup X_s of A_s; similarly the fixer in A_s of the vertex Y_s is the subgroup Y_s. The action of A_s on G_s is faithful because H_s is the fixer of the edge $\langle X_s, Y_s \rangle$ and the only normal subgroup of A_s which is contained in H_s is the trivial subgroup. Note that A_s is transitive on vertices of G_s of valence 3 and also on vertices of valence 2.

Let us define a new graph whose vertex set is A_s/X_s and in which two vertices are adjacent if and only if they are neighbors in G_s of one vertex of valence 2. This new graph is Γ_3, i.e., a regular 3-valent tree. The automorphisms of G_s induce automorphisms of Γ_3 and this establishes an isomorphism between $\mathrm{Aut}(G_s)$ and $\mathrm{Aut}(\Gamma_3)$. Hence, A_s acts on Γ_3 and the action is faithful. The fixer in A_s of the vertex X_s is again the subgroup X_s and the stabilizer of the edge $\{X_s, yX_s\}$ is the subgroup Y_s.

We have the following:

Proposition 11. *The graph whose vertex set is A_s/X_s and in which two vertices uX_s and vX_s are adjacent if and only if $u^{-1}v \in X_s\, yX_s$, is an infinite cubic tree Γ_3. A_s acts on Γ_3 by left multiplication. This action is faithful and A_s is an s-regular subgroup of* $\mathrm{Aut}(\Gamma_3)$. *A_s' is of type s' and A_s'' is of type s''.*

Proof. For the first assertion we need to verify that $u^{-1}v \in X_s\, yX_s$, if and only if uX_s and vX_s are distinct and have a common neighbor in G_s. If $u^{-1}v = x_1\, yx_2$, where x_1, $x_2 \in X$, then uX and vX are both neighbors of ux_1Y_s in G_s. Conversely, let uX_s and vX_s have a common neighbor wY_s in G_s. Then $wy_1 = ux_1$ and $wy_2 = vx_2$ for suitable x_1, $x_2 \in X_s$ and $y_1, y_2 \in Y_s$. Hence $u^{-1}v = x_1 y_1^{-1} y_2 x_2^{-1}$. If $uX_s \neq vX_s$ then $y_1^{-1}y_2 \notin X_s$. But $y_1^{-1}y_2 \in Y_s$ and so $y_1^{-1}y_2 \in yH_s$. Therefore $u^{-1}v \in X_s\, yX_s$.

All the other assertions are obvious from the remarks made above.

A subgroup K of A_s is said to be *small* (in A_s) if $K \lhd A_s$, $K \cap (X_s \cup Y_s) = 1$, and $(A_s : KX_s) > 2$. In that case $Y_s \not\subset KX_s$ and in fact $(A_s : KX_s) \geqslant 4$.

THEOREM 1. *Let G be a connected cubic graph and A an s-regular subgroup of* Aut(G), $1 \leqslant s \leqslant 5$. *Let $A_s = A'_s$ or A''_s be of the same type as A. Then there exists a covering morphism (f, g): $(\Gamma_3, A_s) \to (G, A)$, where* ker g *is small in A_s.*

Conversely, let K be a small subgroup of A_s and put $A_K = A_s/K$. Let G_K be the graph whose vertex-set is A_s/KX_s and in which two vertices uKX_s and vKX_s are adjacent iff $u^{-1}v \in KX_s\, yX_s$. Then G_K is a connected cubic graph, A_K is an s-regular subgroup of Aut(G_K) *and A_K is of the same type as A_s. There is a covering morphism (f, g): $(\Gamma_3, A_s) \to (G_K, A_K)$, where g: $A_s \to A_K$ is the canonical map and f: $\Gamma_3 \to G_K$ is defined by $f(uX) = uKX$.*

Proof. Let $e = \{u, v\}$ be an edge of G. The amalgam $(A(u), A(e))$ is isomorphic to (X_s, Y_s) by Proposition 10. An isomorphism g: $(X_s, Y_s) \to (A(u), A(e))$ can be extended to a unique homomorphism g: $A_s \to A$, and g is surjective by Proposition 1(ii).

If $z \in xX_s$ $(x \in A_s)$ then $z = xx_1$, $x_1 \in X_s$ and $g(z)(u) = g(x)\, g(x_1)(u) = g(x)(u)$ because $g(x_1) \in g(X_s) = A(u)$. Therefore there exists a map f: $A_s/X_s \to G$ such that $f(xX_s) = g(x)(u)$ for all $x \in A_s$. Let z_1X_s and z_2X_s be adjacent in Γ_3. Then $z_1^{-1}z_2 \in X_s\, yX_s$, i.e., $z_1^{-1}z_2 = x_1\, yx_2$, where $x_1, x_2 \in X_s$. We claim that $a = f(z_1X_s) = g(z_1)(u)$ and $b = f(z_2X_s) = g(z_2)(u)$ are adjacent in G.

For this it suffices to show that the vertices $g(z_1^{-1})(b) = g(z_1^{-1}z_2)(u)$ and $g(z_1^{-1})(a) = u$ are adjacent. This follows from

$$g(z_1^{-1}z_2)(u) = g(x_1\, yx_2)(u) = g(x_1)\; g(y)(u) = g(x_1)(v),$$

since $g(x_1) \in A(u)$ and $g(y)$ flips the edge $\{u, v\}$. It follows that g is a graph homomorphism and it is now easy to check that (f, g) is a covering morphism.

Now we pass to the converse. Since $A_s \neq KX_s$ and $A_s = \langle X_s, y\rangle$ we have $y \notin KX_s$. The graph G_K is connected because KX_s and y generate A_s. Since

A_s acts transitively on G_K, the graph G_K is regular. The valency of G_K is equal to the number of left cosets of KX_s in $KX_s yKX_s = KX_s yX_s K$. But this last number is equal to the index of $yX_s yK \cap X_s K$ in $X_s K$. We claim that $yX_s yK \cap X_s K = (X_s \cap Y_s)K$ and consequently G_K is a cubic graph. Otherwise we would have $yK_s yK \cap X_s K \supsetneq (X_s \cap Y_s)K$ and since $(X_s \cap Y_s)K$ is a maximal subgroup of both $X_s K$ and $yX_s yK$, we find that $yX_s yK = X_s K$. This means that y normalizes KX_s and since $y^2 \in X_s$ we obtain $(A_s : KX_s) = 2$, which is again a contradiction.

Thus G_K is a connected cubic graph on which A_s acts. Let N be the kernel of the corresponding homomorphism $A_s \to \operatorname{Aut}(G_K)$. We have $K \subset N$ and since elements of N fix the vertices KX_s and yKX_s of G_K we also have

$$N \subset KX^s \cap KyX_s y = K(X_s \cap Y_s).$$

Therefore $N = K \cdot (N \cap X_s \cap Y_s)$. The group $N \cap X_s \cap Y_s$ is normal in both X_s and Y_s and consequently $N \cap X_s \cap Y_s = \{1\}$, and $N = K$. Hence the group $A_K = A_s/K$ acts on G_K faithfully and we can consider it as a subgroup of $\operatorname{Aut}(G_K)$. Since the amalgam $(KX_s/K, KY_s/K)$ is isomorphic to the amalgam (X_s, Y_s) we infer that A_K is of the same type as A_s.

Now it is easy to verify that (f, g) is a covering morphism.

THEOREM 2. *Let* $A_s = A'_s$ $(1 \leqslant s \leqslant 5)$ resp. $A_s = A''_s$ $(s = 2$ or $4)$ *and let* A *be any s-regular subgroup of* $\operatorname{Aut}(\Gamma_3)$ *of the same type as* A_s. *Then* A_s *and* A *are conjugate in* $\operatorname{Aut}(\Gamma_3)$.

Proof. By Theorem 1 there is a covering morphism (f, g): $(\Gamma_3, A_s) \to (\Gamma_3, A)$. For $x \in A_s$ let $\phi_x : A_s/X_s \to A_s/X_s$ be the left multiplication by x. Then we have $f \circ \phi_x = g(x) \circ f$ for all $x \in A_s$. Since f is necessarily an automorphism of Γ_3, we have $g(x) = f \circ \phi_x \circ f^{-1} = fxf^{-1}$. Hence $g: A_s \to A$ is an isomorphism obtained by restricting the inner automorphism of $\operatorname{Aut}(\Gamma_3)$ induced by f.

PROPOSITION 12. *Let K be a small subgroup of* $A_s = A'_s$ *or* A''_s. *Then* G_K *is bipartite iff* $K \subset A_s^+$.

Proof. Since $(A_K)^+ = A_s^+ K/K$, the claim follows from Proposition 1(iv).

6. REGULAR SUBGROUPS OF REGULAR GROUPS

Let G be a connected cubic graph and A an s-regular subgroup of $\operatorname{Aut}(g)$, $1 \leqslant s \leqslant 5$. Assume that A has a t-regular subgroup B.

THEOREM 3. *Let G, A, B be as above with* $1 \leqslant t < s \leqslant 5$. *The possible types for A and B are indicated by "Yes" in the table below*

$B \backslash A$	$2'$	$2''$	$3'$	$4'$	$4''$	$5'$
$1'$	Yes	No	Yes	Yes	No	Yes
$2'$			Yes	No	No	No
$2''$			Yes	No	No	No
$3'$				No	No	No
$4'$						Yes
$4''$						Yes

Proof. Since the groups A'_s and A''_s are s-regular on Γ_3, the examples for all "Yes" entries are provided by $G = \Gamma_3$. Indeed, the amalgam embeddings $f_{12}, f'_{23}, f''_{23}, f_{14}, f'_{45}$ and f''_{45} (see Section 4) extend to the group embeddings $A'_1 \to A'_2$, $A'_2 \to A'_3$, $A''_2 \to A'_3$, $A'_1 \to A'_4$, $A'_4 \to A'_5$, and $A''_4 \to A'_5$, respectively.

If A is of type $2''$ or $4''$ then B cannot be of type $1'$ because A contains no involutions flipping an edge.

Let S be a 3-arc in G and put $S(0) = x$, $S(1) = y$, $S(2) = z$, $S(3) = u$. Let $A_0(y, z)$ be the subgroup of $A(y, z)$ consisting of those $\alpha \in A(y, z)$ which satisfy $\alpha(x) \neq x \Leftrightarrow \alpha(u) \neq u$. If $s = 4$ then using the notation of Proposition 4 and taking $v_1 = y$, $v_2 = z$ we obtain $A_0(y, z) = \langle \tilde{a}\tilde{c} \rangle$. If $s = 5$ then using the notation of Proposition 5 and taking $y = b$, $z = c$ we obtain $A_0(y, z) = \langle \tilde{a}\tilde{d}, \tilde{b} \rangle$. In both cases we see that if $\alpha \in A_0(y, z)$ and $\alpha(x) \neq x$ then α has order 4.

Let $B_0(y, z) = B \cap A_0(y, z)$. If $t = 2$ then $B_0(y, z) = \langle \alpha \rangle$, where α has order 2 and $\alpha(x) \neq x$. If $t = 3$ then there exists an $\alpha \in B_0(y, z)$ such that $\alpha(x) \neq x$. Since also $\alpha(u) \neq u$, it follows from the 3-regularity of B that $\alpha^2 = 1$. Hence, in both cases there exists an $\alpha \in B_0(y, z)$ such that $\alpha(x) \neq x$ and α has order 2.

Since $B_0(y, z)$ is a subgroup of $A_0(y, z)$ it follows that $(s, t) \neq (5, 2)$, $(5, 3)$, $(4, 2)$, or $(4, 3)$.

We have taken care of all "No" entries in our table.

COROLLARY. *Let G be a connected cubic graph and assume that* Aut(G) *has two subgroups A and B such that A is* 2- *or* 3-*regular and B is* 4- *or* 5-*regular. Then* $G = \Gamma_3$.

Proof. Since Aut(G) is t-transitive but not t-regular it follows that it is $(t + 1)$-transitive. This argument can be repeated ad infinitum and hence

Aut(G) is ω-transitive. Therefore G cannot contain any circuits, i.e., G is a tree.

We now expand on the "Yes" entries in our table.

THEOREM 4. *Let A be an s-regular subgroup of* Aut(Γ_3). *Then we have*:

(i) *If A is of type $2'$, $3'$, or $5'$ then it has precisely two $(s-1)$-regular subgroups, say, B' and B''. If $s=3$ or 5 then B', say, is of the first kind and B'' of the second kind. If (a,α) is a shunt with $\alpha \in A$ then either $\alpha \in B'$ and $\alpha \notin B''$ or $\alpha \in B''$ and $\alpha \notin B'$. Two A-shunts (a,α) and (b,β) are A-conjugate if and only if α and β are both in B' or both in B''.*

(ii) *If A is of type $4'$ then it has precisely 16 1-regular subgroups which split into two conjugacy classes, each of size 8. Let B' resp. B'' be the set-theoretic union of the 1-regular subgroups in the first resp. second conjugacy class. If (a,α) is a shunt with $\alpha \in A$ then either $\alpha \in B'$ and $\alpha \notin B''$ or $\alpha \in B''$ and $\alpha \notin B'$. Two A-shunts (a,α) and (b,β) are A-conjugate if and only if α and β belong both to B' or both to B''.*

Proof. (i) We saw in the proof of Theorem 3 that A_5' contains A_4' and A_4'' as subgroups, A_3' contains A_2' and A_2'' as subgroups, and A_2' contains A_1' as a subgroup. In fact we have two subgroups of A_2' isomorphic to A_1', namely, $\langle ab, y\rangle$ and $\langle ab, yb\rangle$. We denote these two subgroups by B' and B'' for $s=2,3,5$.

Since B' is generated by two shunts it follows that there exists a shunt (a,α) with $\alpha \in B' \backslash B''$. Similarly, there exists a shunt (b,β) with $\beta \in B'' \backslash B'$. Since $(A:B')=(A:B'')=2$ it follows that (a,α) and (b,β) are not A-conjugate. Lemma 2 then implies that if (c,γ) is an A-shunt then either $\gamma \in B' \backslash B''$ or $\gamma \in B'' \backslash B'$. It follows that a B'-shunt and a B''-shunt cannot be A-conjugate. Consequently, any two B'-shunts are conjugate in A and similarly any two B''-shunts are conjugate in A.

Now, let B be any $(s-1)$-regular subgroup of A. If (c,γ) is a B-shunt then (c,γ) is either A-conjugate to (a,α) or to (b,β). In the first case we obtain $B' \subset B$ and in the second case $B'' \subset B$. Thus $B=B'$ or B''.

(ii) We know that A has a 1-regular subgroup, say B_1'. By Proposition 9, B_1' is not normal in A. By Theorem 3, B_1' is a maximal subgroup of A and hence it is its own normalizer in A. Therefore B_1' has 8 conjugates in A which we denote by B_i', $1 \leqslant i \leqslant 8$. By Theorem 3 there is a 5-regular subgroup H of Aut(Γ_3) containing A. Again B_1' is its own normalizer in H since H contains no 2-regular subgroups. Hence B_1' has 16 conjugates in H and all these conjugates lie in A. These conjugates include B_i', $1 \leqslant i \leqslant 8$, and we denote the remaining 8 by B_i'', $1 \leqslant i \leqslant 8$.

Now, let B be any 1-regular subgroup of A. Let us use the notation from Fig. 3. Since $B(v_2) \cong C_3$ and $A(v_2) \cong S_4$ it follows that there are at most four choices for $B(v_2)$. There is an element $\alpha \in B$ of order 2 such that $\alpha(v_2)=v_3$

and $\alpha(v_3) = v_2$. Hence $\alpha \in A(c)$ and using Proposition 4 we find that α must be one of the following: ξ, $\tilde{b}\xi\tilde{b}$, $\tilde{c}\xi$, or $\tilde{c}\tilde{b}\xi\tilde{b}$. Since $B = \langle B(v_2), \alpha\rangle$ by Proposition 1(ii), we see that there are at most 16 1-regular subgroups in A. Hence there are precisely 16 such subgroups.

Let (a, α) be a shunt with $\alpha \in B'_1$. Let (a, β) be a shunt with $\beta \in A$ having overlap 4 with (a, α). Since (a, β) is H-conjugate to (a, α) by (i), we see that every A-shunt (c, γ) is such that γ belongs to a 1-regular subgroup of A. All the remaining assertions will follow when we show that any two B'_1-shunts are A-conjugate. Let (a, α) and (a, β) be two shunts of overlap 1 in, say, B'_1. By Proposition 6, B'_1 is of the second kind. Thus, (a, α) is B'_1-conjugate to (a, β^{-1}). Again by Proposition 6, A'_4 is of the first kind and, therefore, (a, β^{-1}) is A-conjugate to (a, β). So (a, α) is in fact A-conjugate to (a, β).

7. Centralizers and Normalizers

Theorem 5. *Let A be an s-regular subgroup of* $\mathrm{Aut}(\Gamma_3)$, $1 \leqslant s \leqslant 5$. *Then the centralizer of A in* $\mathrm{Aut}(\Gamma_3)$ *is trivial.*

Proof. Let α be a nonidentity element of $\mathrm{Aut}(\Gamma_3)$, and x a vertex of Γ_3 not fixed by α, say, $\alpha(x) = y$. Since A is arc-transitive there exists an element p of order 3 in $A(x)$. Now, p only fixes x since the graph is Γ_3. Since $\alpha p \alpha^{-1}(y) = y$, $\alpha p \alpha^{-1} \neq p$. Thus α is not in the centralizer of A.

Theorem 6. *Let A be an s-regular subgroup of* $\mathrm{Aut}(\Gamma_3)$, $1 \leqslant s \leqslant 5$. *We have*

(i) *if $s = 1$, 2, or 4 then the normalizer of A in* $\mathrm{Aut}(\Gamma_3)$ *is the unique $(s+1)$-regular subgroup containing A;*

(ii) *if $s = 3$ or 5 then A is its own normalizer in* $\mathrm{Aut}(\Gamma_3)$.

Proof. (i) We know that there exists an $(s+1)$-regular subgroup H of $\mathrm{Aut}(\Gamma_3)$ containing A. Let (a, α) and (a, β) be two A-shunts with overlap s. Let S be the $(s+1)$-arc $S(i) = \alpha^i(a)$, $0 \leqslant i \leqslant s+1$. Let $\gamma \in \mathrm{Aut}(\Gamma_3)$ normalize A. We can choose $\delta \in H$ such that $\delta \circ \gamma \circ S = S$. Hence if $\sigma = \delta \circ \gamma$ then $\sigma\alpha\sigma^{-1} = \alpha$ and consequently $\sigma\beta\sigma^{-1} = \beta$. Since $A = \langle \alpha, \beta\rangle$ we conclude that σ centralizes A and so $\sigma = 1$ by Theorem 5. Thus $\gamma = \delta^{-1} \in H$ and so H is the normalizer of A.

(ii) In this case A contains a unique $(s-1)$-regular subgroup B of the first kind. Hence if $\alpha \in \mathrm{Aut}(\Gamma_3)$ normalizes A, it also normalizes B and so (i) implies that $\alpha \in A$.

Proposition 13. *Let B be a 1-regular subgroup of* $\mathrm{Aut}(\Gamma_3)$. *Then there are precisely two 4-regular subgroups (both of the first kind) containing B.*

Proof. By Theorems 2 and 4 we know that there exists a subgroup A of $\mathrm{Aut}(\Gamma_3)$ of type $4'$ containing B. Let H be the normalizer of B in $\mathrm{Aut}(\Gamma_3)$. Then H is 2-regular and $H \cap A = B$. Hence if $\alpha \in H\backslash B$ then $\alpha A \alpha^{-1} \neq A$. We claim that A and $\alpha A \alpha^{-1}$ are the only 4-regular subgroups containing B.

Let K be such a subgroup. By Theorem 3, K is of the first kind. By Theorem 2 there is a $\beta \in \mathrm{Aut}(\Gamma_3)$ such that $\beta K \beta^{-1} = A$. Let L be the normalizer of A in $\mathrm{Aut}(\Gamma_3)$. Then L is 5-regular and by the proof of Theorem 4(ii) there is a $\gamma \in L$ such that $\gamma\beta B \beta^{-1}\gamma^{-1} = B$. Thus $\delta = \gamma\beta \in H$ and $K = \beta^{-1} A \beta = \delta^{-1}\gamma A \gamma^{-1}\delta = \delta^{-1} A \delta$, and so $K = A$ or $\alpha A \alpha^{-1}$.

In Fig. 5 we have indicated the regular subgroups of $\mathrm{Aut}(\Gamma_3)$ which contain a fixed 1-regular subgroup.

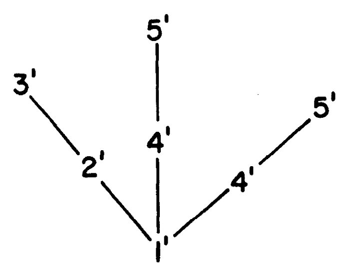

FIGURE 5

Let A be an s-regular subgroup of $\mathrm{Aut}(\Gamma_3)$ and B a t-regular subgroup. Assume that $A \cap B$ is 1-transitive. We claim that if $s = 2$ or 3 and $t = 4$ or 5 then $A \cap B$ is 1-regular. This is clear since otherwise $A \cap B$ would be k-regular with $2 \leqslant k \leqslant 3$ which contradicts Theorem 3. Similarly, if $s = t = 5$ and $A \neq B$ then $A \cap B$ is 1-regular. Otherwise $A \cap B$ would be 4-regular and we would contradict Theorem 6(ii).

8. BIG SUBGROUPS OF A_s

Let $A_s = A_s'$ $(1 \leqslant s \leqslant 5)$ or A_s'' $(s = 2$ or $4)$. A subgroup K of A_s is called *big* (in A_s) if $K \lhd A_s$ and $K \cap (X_s \cup Y_s) \neq \{1\}$.

PROPOSITION 14. *We have* $A_s^+ = X_s *_{H_s} yX_sy^{-1}$ *and* A_s^+ *is the normal closure of* X_s *in* A_s.

Proof. The first assertion follows from [10, Theorem 6, pp. 1–49]. The second is then obvious.

PROPOSITION 15. *The big subgroups of* A_1' *are* A_1', $(A_1')^+$, *and* $E_1 = \langle y, aya^{-1}, a^{-1}ya \rangle$. *We have* $(A_1' : E_1) = 3$ *and* E_1 *is the normal closure of* $\langle y \rangle$.

Proof. If K is a big subgroup of A'_1 then either $K \supset X_1$ or $K \supset Y_1$ and all the claims easily follow.

PROPOSITION 16. *The big subgroups of A'_2 are the normal closures of $\langle y\rangle$ and $\langle yb\rangle$, and A'_2, $(A'_2)^+$, $E_2=\langle ab, yaby\rangle$, $B'=\langle ab, y\rangle$, $B''=\langle ab, yb\rangle$.*

Proof. Let K be a big subgroup of A'_2. If $K\cap X_2\neq\{1\}$ then $K\supset\langle ab\rangle$ and consequently $K\supset E_2$. Since $E_2\lhd A'_2$ and $A'_2/E_2\cong C_2\times C_2$ it is clear that K is one of A'_2, $(A'_2)^+$, B', B'', or E_2. Note that $E_2=B'\cap B''=B'\cap(A'_2)^+=B''\cap(A'_2)^+$. If $K\cap X_2=\{1\}$ then $K\cap Y_2=\langle y\rangle$ or $\langle yb\rangle$ and then K must be the normal closure of $\langle y\rangle$ or $\langle yb\rangle$ in A'_2.

PROPOSITION 17. *The big subgroups of $A_2=A''_2$ are A_2, A_2^+, and $E_2=\langle ab, yaby^{-1}\rangle$.*

Proof. If $K\lhd A_2$ is big then $U=K\cap X_2\neq\{1\}$ because $Y_2=\langle y\rangle\cong C_4$ and $y^2=b\in X_2$. Since $U\lhd X_2$ we have $U\supset\langle ab\rangle$ and $K\supset E_2$. It remains to note that $A''_2/E_2\cong C_4$.

PROPOSITION 18. *The big subgroups of A'_3 are A'_3, $(A'_3)^+$, $A'_2=\langle ab, bc, y\rangle$, $A''_2=\langle ab, bc, yb\rangle$, $A'_2\cap A''_2=(A'_2)^+=(A''_2)^+$, and $E_2=\langle ac, yacy\rangle$.*

Proof. Let K be a subgroup of A'_3. Since every non-trivial normal subgroup of $Y_3\cong D_4$ contains the central element bc it follows that $U=K\cap X_3\neq\{1\}$. Since also $U\lhd X_3=\langle a, c\rangle\times\langle b\rangle$, $\langle a, c\rangle\cong D_3$, $\langle b\rangle\cong C_2$ we must have either $ac\in U$ or $b\in U$. Since a, b, c are all conjugate in A'_3, we have in both cases $ac\in U$ and so $K\supset E_2$. Our claim now follows from the fact that $E_2\lhd A'_3$ and $A'_3/E_2\cong D_4$.

PROPOSITION 19. *The big subgroups of $A_4=A'_4$ or A''_4 are A_4 and A_4^+.*

Proof. Let K be a big subgroup of A_4. We must have again $U=K\cap X_4\neq\{1\}$. We need only show that $U=X_4$. Assume the contrary. Then since $U\lhd X_4\cong S_4$ we must have $U=\langle b,c\rangle$. Since a, b, c, d are all conjugate in A_4 we obtain a contradiction.

PROPOSITION 20. *The big subgroups of A'_5 are A'_5, $(A'_5)^+$, $A'_4=\langle ab, bc, cd, de, y\rangle$, $A''_4=\langle ab, bc, cd, de, yc\rangle$, and $A'_4\cap A''_4=(A')^+=(A''_4)^+$.*

Proof. Let K be a big subgroup of A'_5. Since the center of Y_5 lies in X_5 we must have $U=K\cap X_5\neq\{1\}$. Recall that $X_5\cong S_4\times C_2$ and that $\langle c\rangle$ is the center of X_5. It is easy to check that X_5 admits only two direct decompositions.

$$X_5 = \langle ab, bc, cd, de \rangle \times \langle c \rangle,$$
$$X_5 = \langle a, bc, cd, e \rangle \times \langle c \rangle.$$

We cannot have $U \subset \langle b, c, d \rangle$ because a, b, c, d, e are all conjugate in A'_5. Since $U \lhd X_5$ it follows that U is one of the following: X_5, $\langle ab, bc, cd, de \rangle$, or $\langle a, bc, cd, e \rangle$. The last case is impossible since a, b, c, d, e are conjugate. Hence U contains the normal closure of $\langle ab, bc, cd, de \rangle$ in A'_5 which coincides with $A'_4 \cap A''_4$. All the claims now follow easily.

9. Classification Problem

An *s'-object* $(1 \leqslant s \leqslant 5)$ resp. an *s''-object* $(s = 2$ or $4)$ is an ordered pair (G, A), where G is a connected cubic graph and A is a subgroup of $\mathrm{Aut}(G)$ of type s' resp. s''. An *s-object* is either an s'-object or an s''-object. We shall also need more special objects. For instance, a $(4', 1')$-object is an ordered triple (G, A, B), where G is as above, A is a subgroup of $\mathrm{Aut}(G)$ of type $4'$ and B is a subgroup of A of type $1'$.

An s'-object (G, A) is *minimal* if every covering morphism (f, g): $(G, A) \to (G', A')$, where (G', A') is also an s'-object, is in fact an isomorphism, i.e., both f: $G \to G'$ and g: $A \to A'$ are isomorphisms. Minimal objects of other types are defined similarly.

If K is a small subgroup of A'_s resp. A''_s then we can associate to K an s'-object resp. s''-object (G_K, A_K) as described in Theorem 1. Moreover every s'-object resp. s''-object is obtained in this way (up to isomorphism). We shall show in this section that if (G, A) is an s'-object resp. s''-object then there is either one or two small subgroups K of A'_s resp. A''_s such that $(G_K, A_K) \cong (G, A)$. Hence the problem of classification of s'-objects resp. s''-object reduces to the problem of classification of normal subgroups of A'_s resp. A''_s.

Lemma 4. *Let K_1, K_2 be small subgroups of $A_s = A'_s$ or A''_s such that $K_1 \leqslant K_2$. Then there is a natural covering morphism (f, g): $(G_{K_1}, A_{K_1}) \to (G_{K_2}, A_{K_2})$.*

Proof. Define g: $A_{K_1} \to A_{K_2}$ to be the canonical map $A_s/K_1 \to A_s/K_2$, and define f: $G_{K_1} \to G_{K_2}$ to be the canonical map from A_s/K_1X_s to A_s/K_2X_s. Then it is easy to check that (f, g) is a covering morphism.

It is clear from this lemma that an s'-object resp. s''-object (G_K, A_K) is minimal if and only if K is a maximal small subgroup of $A_s = A'_s$ resp. A''_s. One can use Zorn's lemma to show that every small subgroup of A_s is contained in a maximal small subgroup of A_s. We shall say that a small subgroup K of A_s is *even* or *odd* according to whether $K \subset A_s^+$ or $K \not\subset A_s^+$.

We shall say that a graph G is an s'-object or s''-object if $(G, \mathrm{Aut}(G))$ is such an object.

If (f, g): $(G, A) \to (\bar{G}, \bar{A})$ is a covering morphism we define $\ker(f, g) = \ker g$.

LEMMA 5. *Let (f, g): $(G, A) \to (\bar{G}, \bar{A})$ be a covering morphism and $h \in \bar{A}$. Then (hf, hgh^{-1}): $(G, A) \to (\bar{G}, \bar{A})$ is also a covering morphism, where hgh^{-1}: $A \to \bar{A}$ is defined by $(hgh^{-1})(\alpha) = hg(\alpha)h^{-1}$.*

Proof. It is clear that $hf: G \to \bar{G}$ is a graph covering and that $hgh^{-1}: A \to \bar{A}$ is a group homomorphism. For $\alpha \in A$ we have $g(\alpha)f = f\alpha$ and consequently $(hgh^{-1})(\alpha) \circ hf = hg(\alpha)h^{-1}\, hf = hg(\alpha)f = hf\alpha$. Thus the lemma is proved.

THEOREM 7. *Let (G, A) be an s'-object, $s = 3$ or 5. Then all covering morphisms (f, g): $(\Gamma_3, A_s) \to (G, A)$ have the same kernel.*

Proof. Let (f, g) and (f', g') be two such covering morphisms. Further let S be an s-arc in Γ_3. Now $f \circ S$ and $f' \circ S$ are two s-arc in G. Since A is s-transitive there exists an $h \in A$ such that $h \circ f \circ S = f' \circ S$. By Lemma 5 (hf, hgh^{-1}) is also a covering morphism having the same kernel as (f, g) and hf and f' coincide on $S(i)$, $0 \leqslant i \leqslant s$.

Hence without loss of generality we may assume that $f \circ S = f' \circ S$. It suffices to show that in this case $g = g'$.

If x is a vertex of Γ_3 we let $\tilde{x} \in A_s$ be the involution defined in Proposition 3 or 5 and similarly for y a vertex of G. It is clear from these propositions that if $f(x) = y$ then $g(\tilde{x}) = \tilde{y}$. Similarly, if $\alpha \in A$ is the unique involution such that $\alpha \circ S = S'$ is the opposite s-arc of S then $g(\alpha) \in A$ is the unique involution such that $g(\alpha) \circ f \circ S$ is the opposite s-arc of $f \circ S$. Hence, it follows that $g(\tilde{x}) = g'(\tilde{x})$ for $x = S(i)$ $(0 \leqslant i \leqslant x)$ and $g(\alpha) = g'(\alpha)$. Since α and $\tilde{x}$, for $x = S(i)$ $(0 \leqslant i \leqslant s)$, together generate A_s, it follows that $g = g'$ and the proof is completed.

LEMMA 6. *Let (f, g): $(G, A) \to (G', A')$ be a covering morphism and let $h \in \mathrm{Aut}(G)$ be in the normalizer of A. Then (fh, g^h): $(G, A) \to (G', A')$ is also a covering morphism, where $g^h(\alpha) = g(h\alpha h^{-1})$ for $\alpha \in A$.*

Proof. This follows from $g^h(\alpha)fh = g(h\alpha h^{-1})fh = fh\alpha h^{-1}h = fh\alpha$.

LEMMA 7. *Let (f, g): $(G, A) \to (G', A')$ be a covering morphism and A, A' be s-regular groups. Then $f(x) = f(y)$ iff there exists $\alpha \in \ker g$ such that $\alpha(x) = y$.*

Proof. Let $y = \alpha(x)$ where $\alpha \in \ker g$. Then

$$f(y) = f(\alpha(x)) = (f \circ \alpha)(x) = (g(\alpha) \circ f)(x) = f(x).$$

Conversely, let x and y be vertices of G such that $f(x)=f(y)$. We can choose s-arcs S_1 and S_2 such that $S_1(0)=x$, $S_2(0)=y$, and $f\circ S_1=f\circ S_2$. Since A is s-transitive there exists $\alpha\in A$ such that $\alpha\circ S_1=S_2$. Then $g(\alpha)\in A'$ fixes the s-arc $S=f\circ S_1$ in G' and so $g(\alpha)=1$ by s-regularity of A'. Thus $\alpha\in\ker g$.

THEOREM 8. *Let $A_s=A'_s$, $s=1, 2, 4$ or $A_s=A''_s$, $s=2, 4$. Let (G,A) be an object of the same type as (Γ_3, A_s). There are either one or two subgroups of G which are kernels for the covering morphisms from (Γ_3, A_3) to (G, A). If there is exactly one then G is $(s+1)$-transitive.*

Proof. Let S and T be fixed s-arcs in Γ_3 and G, respectively. By Lemma 5 we can consider only covering morphisms (f, g): $(\Gamma_3, A_3)\to(G,A)$ such that $f\circ S=T$.

Let $S(0)=x$, $T(0)=y$ and let (x,α), (x,β), (y,γ), (y,δ) be shunts such that α, $\beta\in A_s$; γ, $\delta\in A$, $\alpha\neq\beta$, $\gamma\neq\delta$, $\alpha(S(i))=\beta(S(i))=S(i+1)$, $\gamma(S(i))=\delta(S(i))=S(i+1)$ for $0\leqslant i<s$. Then we must have $\{g(\alpha), g(\beta)\}=\{\gamma,\delta\}$ and hence either $g(\alpha)=\gamma$, $g(\beta)=\delta$ or $g(\alpha)=\delta$, $g(\beta)=\gamma$. Since $A=\langle\alpha,\beta\rangle$ we conclude that there are at most two possibilities for $\ker g$. We know that at least one such covering morphism exists (Theorem 1).

Assume now that there is only one such kernel, say $\ker g=K$. Note that $A_s\leqslant A'_{s+1}$ and in fact A'_{s+1} is the normalizer of A_s in $\mathrm{Aut}(\Gamma_3)$ (Theorem 6.) The uniqueness of K implies that $K\lhd A'_{s+1}$ and consequently G is $(s+1)$-transitive.

PROPOSITION 21. *Let (G,A) be a $1'$-object, $e=\{u,v\}$ an edge of G, $A(u)=\langle p\rangle\cong C_3$ and $A(e)=\langle\delta\rangle\cong C_2$. The following are equivalent:*

(i) *There is a subgroup B of* $\mathrm{Aut}(G)$ *of type $2'$ containing A;*
(ii) *There is an automorphism ϕ of A such that $\phi(\rho)=\rho^{-1}$, $\phi(\sigma)=\sigma$.*

Proof. (i) $\Rightarrow$ (ii). Let α be the generator of $B(u,v)\cong C_2$. Take ϕ to be the restriction to A of the inner automorphism of B induced by α.

(ii) $\Rightarrow$ (i). If α, $\beta\in A$ and $\alpha(u)=\beta(u)$ then $\alpha^{-1}\beta\in A(u)$ and $\phi(\alpha^{-1}\beta)=\gamma\in A(u)$ so that $\phi(\beta)(u)=\phi(\alpha)\,\gamma(u)=\phi(\alpha)(u)$. Therefore there exists a map f: $G\to G$ such that $f(\alpha(u))=\phi(\alpha)(u)$ for all $\alpha\in A$. If a is a vertex of G and $a=\alpha(u)$ then we have $f(\beta(a))=f(\beta\alpha(u))=\phi(\beta\alpha)(u)=\phi(\beta)$ $\phi(\alpha)(u)=(\beta)$ $(f(a))$, i.e., $f\circ\beta=\phi(\beta)\circ f$ for all $\beta\in A$. It follows that f is surjective. It is also injective because $f(\alpha(u))=f(\beta(u))$ implies $\phi(\alpha)(u)=\phi(\beta)(u)$, $\phi(\alpha^{-1}\beta)\in A(u)$ and consequently $\alpha^{-1}\beta\in A(u)$, $\alpha(u)=\beta(u)$.

If $\{a,b\}$ is an edge of G then there exists $\alpha\in A$ such that $\alpha(u)=a$, $\alpha(v)=b$ and hence $f(a)=f(\alpha(u))=\phi(\alpha)(u)$, $f(b)=f(\alpha(v))=\phi(\alpha)(v)$. Thus $\{f(a), f((b)\}$ is also an edge of G and consequently $f\in\mathrm{Aut}(G)$.

We have also $f^2 = 1$ and $f \circ \alpha \circ f^{-1} = \phi(\alpha)$ for all $\alpha \in A$. Since $f \neq 1$ and $f(u) = u$, $f(v) = f(y(u)) = \phi(y)(u) = y(u) = v$ we have $f \notin A$. Hence $B = \langle A, f \rangle$ is a 2-regular subgroup of Aut(G) and it must be of type $2'$ since $B \supset A$.

PROPOSITION 22. *Let (G, A) be a $2'$-object, resp., $2''$-object and $a = \{u, v\}$, $b = \{v, w\}$ two distinct edges of G. Let $y \in A$ be an element which flips the edge b. Then the following are equivalent:*

(i) *There is a subgroup B of* Aut(G) *of type $3'$ which contains A;*
(ii) *There is an automorphism ϕ of A such that $\phi(\tilde{a}) = \tilde{a}$,*

$$\phi(\tilde{b}) = \tilde{b} \qquad \textit{and} \qquad \phi(y) = y\tilde{b} \textit{ resp., } \phi(y) = y^{-1}.$$

Proof. (i) $\Rightarrow$ (ii). Let α be the generator of $B(u, v, w) \cong C_2$. Then we can take ϕ to be the restriction to A of the inner automorphism of B induced by α.

(ii) $\Rightarrow$ (i). There is a unique map f: $G \to G$ such that $f(\alpha(v)) = \phi(\alpha)(v)$ for all $\alpha \in A$. This f satisfies $f \circ \alpha = \phi(\alpha) \circ f$ for all $\alpha \in A$. It is easy to check that $f \in$ Aut(G), $f^2 = 1$, $f \neq 1$. Since f fixes u, v, w we have $f \notin A$ and hence $B = \langle A, f \rangle$ is 3-regular.

PROPOSITION 23. *Let (G, A) be a $4'$-object, resp., $4''$-object and let $a = \{v_0, v_1\}$, $b = \{v_1, v_2\}$, $c = \{v_2, v_3\}$, $d = \{v_3, v_4\}$ be distinct edges of G. Let $y \in A$ be an element of order* 2 *resp.,* 4 *such that $y(v_1) = v_4$, $y(v_2) = v_3$ and $y(v_4) = v_1$. The following are equivalent:*

(i) *There is a subgroup B of* Aut(G) *of type $5'$ containing A;*
(ii) *There is an automorphism ϕ of A such that $\phi(\tilde{a}) = \tilde{a}$, $\phi(\tilde{b}) = \tilde{b}$, $\phi(\tilde{c}) = \tilde{c}$, $\phi(\tilde{d}) = \tilde{d}$ and $\phi(y) = y\tilde{c}$ resp., $\phi(y) = y^{-1}$.*

Proof. (i) $\Rightarrow$ (ii). Let α be the generator of $B(v_0, v_2, v_4) \cong C_2$. Then we can take ϕ to be the restriction to A of the inner automorphism of B induced by α.

(ii) $\Rightarrow$ (i). There is a unique map f: $G \to G$ such that $f(\alpha(v_2)) = \phi(\alpha)(v_2)$ holds for all $\alpha \in A$. It follows that $f \circ \alpha = \phi(\alpha) \circ f$ for all $\alpha \in A$. Again, it is easy to verify that f is an automorphism G of order 2. Since f fixes the vertices v_i, $0 \leqslant i \leqslant 4$ and $f \neq 1$ it follows that $f \notin A$. Thus $B = \langle A, f \rangle$ is 5-regular.

10. FINITE PRIMITIVE OBJECTS

An s-object (G, A) is said to be primitive if A acts primitively on the vertex-set of G. A graph G is called *vertex primitive* if Aut(G) acts primitively on the vertex set of G.

We shall list here all finite primitive s-objects $(1 \leqslant s \leqslant 5)$.

Note that if (G, A) is a finite primitive, say, $2'$-object then G is a vertex-primitive graph. Since all finite cubic vertex-primitive graphs are known [16] it is easy to deduce the following facts.

PROPOSITION 24. *There is only one finite primitive* $1'$-*object*: (K_4, A_4), *where* A_4 *stands for the alternating group on four letters.*

There are precisely two finite primitive $2'$-*objects*: (K_4, S_4) *and* (P, A_5), *where* P *is the Petersen graph and* A_5 *the alternating group.*

There is only one finite primitive $2''$-*object*: $(G(28), PSL_2(7))$.

There are precisely two finite primitive $3'$-*objects*: (P, S_5) *and* $(G(28), PGL_2(7))$.

The finite primitive $4'$-*objects form an infinite series* $(G_p, PSL_2(p))$ *where* p *is a prime congruent to* $\pm 1 \pmod{16}$.

There is only one finite primitive $4''$-*object*: $(G(234), SL_3(3))$.

There is only one finite primitive $5'$-*object*: $(G(234), \mathrm{Aut}(SL_3(3)))$.

For the definition of the graphs $G(28)$, $G(234)$, *and* G_p *we refer to* [1, p. 125; 2, 16].

11. CONNECTION WITH THE MODULAR GROUP

It is well known that the modular group $PSL_2(\mathbf{Z}) \cong C_3 * C_2$, see [9, p. 139].

PROPOSITION 25. *We have* $A_1' \cong PSL_2(\mathbf{Z})$, $A_2' \cong PGL_2(\mathbf{Z})$, *and* $A_3' \cong \mathrm{Aut}\, PGL_2(\mathbf{Z})$.

Proof. The first claim follows from the above remark and the definition of A_1'.

Let $a' = \left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$, $b' = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $y' = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, considered as elements of $PGL_2(\mathbf{Z})$. Then it is easy to check that the amalgam $(\langle a', b'\rangle, \langle b', y'\rangle)$ is isomorphic to Amalgam $2'$. Hence there exists a homomorphism f: $A_2' \to PGL_2(\mathbf{Z})$ such that $f(a) = a'$, $f(b) = b'$, and $f(y) = y'$. Since $PGL_2(\mathbf{Z})$ is generated by a', b', y' the homomorphism f is surjective. The subgroup $\langle ab, y\rangle = \langle ab\rangle * \langle y\rangle \cong C_3 * C_2$ of A_2' is mapped by f onto $PSL_2(\mathbf{Z})$. Since $f(ab) = a'b'$, $f(y) = y'$ and $PSL_2(\mathbf{Z}) = \langle a'b'\rangle * \langle y'\rangle$ it follows that $\ker f \cap \langle ab, y\rangle = \{1\}$. Since the index of $\langle ab, y\rangle$ in A_2' is 2 and $\langle ab, y\rangle$ is not a direct factor of A_2' it follows that f is injective and consequently an isomorphism.

There is an automorphism α of A_2' such that $\alpha(a) = a$, $\alpha(b) = b$, $\alpha(y) = by$. We have $\alpha^2 = 1$ and we claim that $\mathrm{Aut}(A_2') = A_2' \rtimes \langle\alpha\rangle$. Since A_2' has trivial center we are allowed to consider A_2' as a normal subgroup of $\mathrm{Aut}(A_2')$. Our claim will follow if we prove that $\mathrm{Aut}(A_2') = A_2' \cup \alpha \circ A_2'$. Let $\beta \in \mathrm{Aut}(A_2')$. Then $\beta(X_2) \cong X_2 \cong D_3$ and by the Kurosh subgroup theorem there exists an inner automorphism γ such that $\gamma\beta(X_2) = X_2$. Since $X_2 \cong D_3$ and all three involutions of D_3 are conjugate to each other, there exists an inner automorphism δ of A_2' such that $\delta\gamma\beta(X_2) = X_2$ and $\delta\gamma\beta(b) = b$. Then $\delta\gamma\beta(a)$ is either a or $aba = bab$. In any case there is an inner automorphism ε of A_2' such that $\varepsilon(X_2) = X_2$, $\varepsilon(b) = b$ and $\varepsilon\delta\gamma\beta(a) = a$. Thus $\varepsilon\delta\gamma\beta$ fixes every element of X_2. Since Y_2' is the centralizer of b, we have $\varepsilon\delta\gamma\beta(Y_2') = Y_2'$. Thus either $\varepsilon\delta\gamma\beta(y) = y$ or $\varepsilon\delta\gamma\beta(y) = yb$. In the former case we have $\varepsilon\delta\gamma\beta = 1$ and in the latter $\varepsilon\delta\gamma\beta = \alpha$. Hence our claim about $\mathrm{Aut}(A_2')$ is proved.

Since $A_2' \leqslant A_3'$ and $(A_3' : A_2') = 2$ we have a homomorphism $g: A_3' \to \mathrm{Aut}(A_2')$ induced by conjugation. Since the centralizer of A_2' in A_3' is trivial, this is an embedding. Since $\mathrm{Aut}(A_2') = A_2' \rtimes \langle \alpha \rangle$, g is an isomorphism.

12. Some Subgroups of A_3'

We now describe the subgroups of A_3' shown in Fig. 6. The numbers on this figure are the indices of various subgroups. B' and B'' are the two 1-regular subgroups of A_2', $E_3 = (A_2')^+$ and $E_2 = B' \cap B''$. The subgroup E' (resp., E'') is the unique normal subgroup of B' (resp. B'') of index 3. We know that B' and B'' are conjugate in A_3' as are E' and E''. Note that A_2' has no normal subgroups of index 3 because every homomorphism $X_2 \to C_3$ or $Y_2' \to C_3$ is trivial. Hence $A_2'/E' \cong A_2'/E'' \cong D_3$ and $E'E'' = A_2'$. If we put $T = E' \cap E''$ then $E'/T \cong D_3$. The group $U' = E' \cap E_2$ (resp. $U'' = E'' \cap E_2$) is the commutator subgroup of B' (resp. B'') and $B'/U' \cong B''/U'' \cong C_6$ [9, p. 141].

The group E_2/T is elementary abelian of order 9 and A_3' acts on it as a cyclic group of order 2. Therefore there are precisely two subgroups V, W such that $E_2 \supset V \supset T$, $E_2 \supset W \supset T$, $V/T \cong W/T \cong C_3$, $V \lhd A_3'$, and $W \lhd A_3'$. The subgroups U', U'' are normal in A_2' but they are neither big nor small in A_2'. The $3'$-object which is associated to $T \lhd A_3'$ is bipartite and has six vertices. It is easy to see that this object must be $(K_{3,3}, \mathrm{Aut}(K_{3,3}))$.

K_4' and K_4'' are subgroups which correspond to the $2'$-object (K_4, S_4). Next we put $Q' = K_4' \cap E_2$, and $Q'' = K_4'' \cap E_2$. Then Q' and Q'' are the normal subgroups of A_2' which correspond to the graph of the cube. The product $L = K_4'K_4''$ gives rise to the normal subgroup L/K_4' of $A_2'/K_4' \cong S_4$. If $L \neq A_2'$ then we would have $L \subset B'$ since $B'/K_4' \cong A_4$ and every proper normal subgroup of S_4 is contained in A_4 (the alternating subgroup). But $L \subset B'$ is impossible since $K_4'' \subset B'$. Thus we have $K_4'K_4'' = A_2'$, and if $N = K_4' \cap K_4''$ then

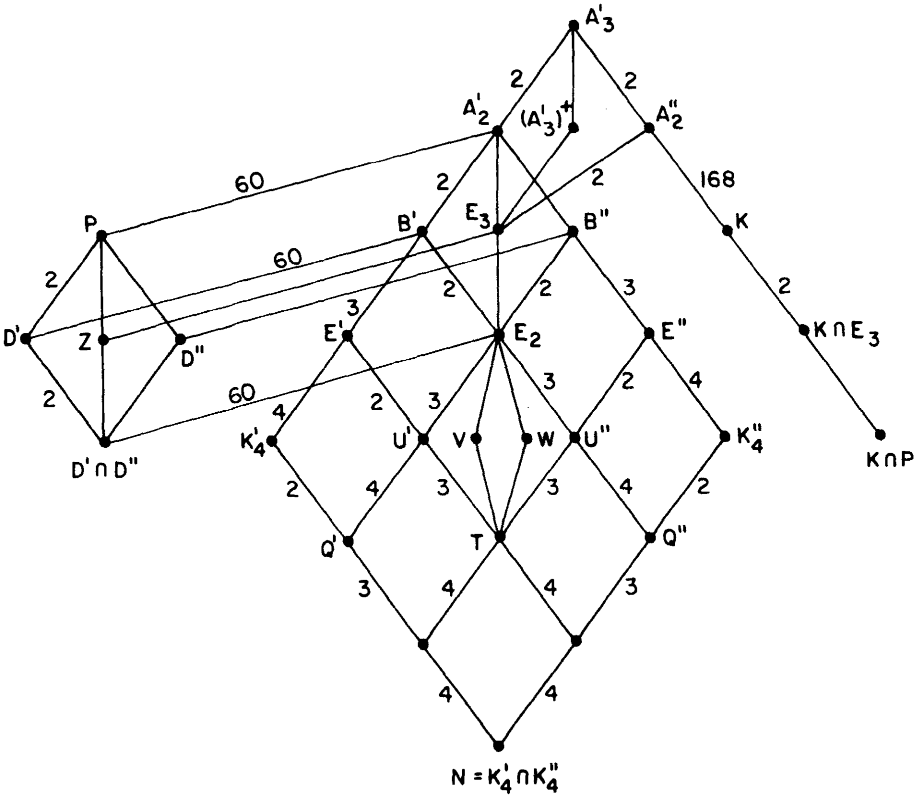

FIGURE 6

$K'_4/N \cong K''_4/N \cong S_4$. We also have $Q'/N \cong Q''/N \cong A_4$ (the alternating group).

We have $T/N \cong ((T \cap Q')/N) \times ((T \cap Q'')/N)$ and so T/N is the elementary abelian group of order 16. Moreover, we have $A'_2/N = (K'_4/N) \times (K''_4/N) = S_4 \times S_4$. It follows that T/N is simple as an A'_3-module, i.e., there are no normal subgroups of A'_3 which lie strictly between N and T. The graph corresponding to $N \lhd A'_3$ is 3-regular; it is a 12-fold cover of the cube and a 16-fold cover of $K_{3,3}$.

The subgroup $P \lhd A'_3$ corresponds to the Petersen graph; D' and D'' are the two normal subgroups of A'_2 which correspond to the dodecahedron, and Z corresponds to Desargues' graph which is 3-regular.

Finally $K \lhd A'_3$ corresponds to the vertex primitive 3-regular graph $G(28)$.

PROPOSITION 26. *Let (G, A) be a $3'$-object.* (1) *If A contains a 1-regular subgroup then G is bipartite and A contains two 2-regular subgroups and two 1-regular subgroups.* (2) *If A contains two 2-regular subgroups then G is bipartite.*

Proof. Let $(f, g): (\Gamma_3, A_3') \to (G, A)$ be a covering morphism. The inverse image of A_3' of a 1-regular (resp. 2-regular) subgroup of A is B' or B'' (resp. A_2' or A_2''). Thus if $K = \ker g$ then $K \subset B'$ or $K \subset B''$ (resp. $K \subset A_2' \cap A_2''$). If $K \subset B'$ then also $K \subset B''$ since B' and B'' are conjugate in A_3'. Thus it follows that $K \subset E_2$ (resp. E_3) and so G is bipartite by Proposition 12. For the case $K \subset E_2$, we have $K \subset B'$, B'', A_2', and A_2''. Thus, by Theorem 1, the images of these four subgroups under g act 1-regularly (resp. 2-regularly) in G.

13. Some Infinite Classes of 2-Regular Cubic Graphs

In this section p will denote a prime $\geqslant 5$. All the matrices stand for the corresponding elements of $PGL_2(p)$.

Let

$$a = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since

$$aby = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \qquad yab = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix},$$

it follows from a theorem of Dickson [6, p. 44] that $\langle ab, y \rangle = PSL_2(p)$.

Note that we have

$$a^2 = b^2 = y^2 = (ab)^2 = (by)^2 = (ab)^3 = 1.$$

We define the graph G_p' as follows: its vertices are the cosets $u\langle ab \rangle$, $u \in PSL_2(p)$ and the edges are $\{u\langle ab \rangle, v\langle ab \rangle\}$ for u, $v \in PSL_2(p)$ and $u^{-1}v \in \langle ab \rangle\, y \langle ab \rangle$.

Proposition 27. *The graph G_p' is a 2-regular cubic non-bipartite graph. We have* $\mathrm{Aut}(G_p') \cong C_2 \times PSL_2(p)$ *according to whether -1 is a square or not* mod. p.

Proof. It is clear that $(G_p', PSL_2(p))$ is a $1'$-object. Since $PSL_2(p)$ is simple, it has no subgroups of index 2. Hence G_p' is not bipartite.

$PSL_2(p) = \langle ab, y \rangle$ admits an automorphism ϕ such that $\phi(ab) = ba = (ab)^{-1}$ and $\phi(y) = y$. Indeed ϕ is the restriction to $PSL_2(p)$ of the inner automorphism of $PGL_2(p)$ induced by b. We have $b \in PSL_2(p)$ precisely when -1 is a square mod p.

From Proposition 21 and its proof it follows that $\mathrm{Aut}(G_p')$ has a subgroup

A which contains $PSL_2(p)$ and $A \cong C_2 \times PSL_2(p)$ or $A \cong PGL_2(p)$ according to whether -1 is a square or not modulo p.

In both cases $PSL_2(p) \lhd \mathrm{Aut}(G'_p)$ and so Proposition 9 implies that $A = \mathrm{Aut}(G'_p)$.

Now let $A = \langle a, b, y\rangle$. Since $\langle ab, y\rangle = PSL_2(p)$ we have either $A = PSL_2(p)$ or $A = PGL_2(p)$. The first case occurs if -1 is a square modulo p, i.e., if $p \equiv 1 \pmod 4$, and the second case otherwise.

We define a graph G''_p as follows: its vertices are the cosets $u\langle a, b\rangle$, $u \in A$ and its edges are $\{u\langle a, b\rangle, v\langle a, b\rangle\}$ for u, $v \in A$ and $u^{-1}v \in \langle a, b\rangle\, y\langle a, b\rangle$. Since the amalgam $(\langle a, b\rangle, \langle b, y\rangle)$ is isomorphic to (X_2, Y'_2) it is clear that (G''_p, A) is a $2'$-object.

PROPOSITION 28. *The graph G''_p is cubic and non-bipartite. If $p > 5$ then G''_p is 2-regular and* $\mathrm{Aut}(G''_p) = A$, *while G''_5 is the Petersen graph.*

Proof. If $A = PSL_2(p)$ then G''_p is not bipartite because A has no subgroups of index 2. If $A = PGL_2(p)$ then, by Proposition 1 (iv), G''_p is not bipartite since the only subgroup of A having index two is $PSL_2(p)$, which does not contain the vertex-fixer $\langle a, b\rangle$.

If $A = PGL_2(p)$ then the subgroup $PSL_2(p)$ is 1-regular. Since every $(3', 1')$-object is bipartite (Proposition 26) it follows that G''_p is not 3-regular. Hence by Theorem 3, G''_p is 2-regular in this case.

Now assume that $A = PSL_2(p)$, i.e., $p \equiv 1 \pmod 4$. Thus $a, b \in PSL_2(p)$.

By Proposition 21 we have to show that when $p > 5$, $PSL_2(p)$ has no automorphisms ϕ such that $\phi(a) = a$, $\phi(b) = b$, and $\phi(y) = by$.

It is known [3, Chap. 4] that the automorphism group of $PSL_2(p)$ is isomorphic to $PGL_2(p)$. In fact every element of $PGL_2(p)$ induces by conjugation an automorphism of $PSL_2(p)$ and this gives us an isomorphism $PGL_2(p) \to \mathrm{Aut}(PSL_2(p))$. Assume that ϕ exists and is induced by $z = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$. Since $zb = bz$ we find that

$$\lambda \begin{pmatrix}\beta & \alpha\\ \delta & \gamma\end{pmatrix} = \begin{pmatrix}\gamma & \delta\\ \alpha & \beta\end{pmatrix}$$

for some λ. Thus $\gamma = \lambda\beta$, $\beta = \lambda\gamma$, $(\lambda^2 - 1)\beta = 0$. We cannot have $\beta = 0$ since then also $\gamma = 0$ and $zy \neq byz$. Therefore $\beta \neq 0$ and $\lambda = \pm 1$. Thus

$$z = \begin{pmatrix}\alpha & \beta\\ \beta & \alpha\end{pmatrix} \quad \text{or} \quad z = \begin{pmatrix}\alpha & \beta\\ -\beta & -\alpha\end{pmatrix}.$$

If $z = \left(\begin{smallmatrix}\alpha & \beta\\ \beta & \alpha\end{smallmatrix}\right)$ then $zy = byz$ gives

$$\mu \begin{pmatrix}-\beta & \alpha\\ -\alpha & \beta\end{pmatrix} = \begin{pmatrix}-\alpha & -\beta\\ \beta & \alpha\end{pmatrix}$$

for some μ. Hence $\alpha = \mu\beta$, $\beta = -\mu\alpha$ and so $\mu^2 = -1$ and $z = \left(\begin{smallmatrix}1 & \mu\\ \mu & 1\end{smallmatrix}\right)$.

The equation $za = az$ now forces $2\mu = -1$. Thus $1 = 4\mu^2 = -4 \pmod p$ and so $p = 5$.

If $z = \left(\begin{smallmatrix} \alpha & \beta \\ -\beta & -\alpha \end{smallmatrix}\right)$ then $zy = byz$ gives

$$\mu \begin{pmatrix} -\beta & \alpha \\ \alpha & -\beta \end{pmatrix} = \begin{pmatrix} -\alpha & -\beta \\ -\beta & -\alpha \end{pmatrix}$$

for some μ. Again $\alpha = \mu\beta$, $\beta = -\mu\alpha$, $\mu^2 = -1$ and so $z = \left(\begin{smallmatrix} 1 & -\mu \\ \mu & -1 \end{smallmatrix}\right)$.

The equation $za = az$ now forces $\mu = 0$ which is a contradiction.

We leave it to the reader to verify that in case when -1 is a square modulo p then we have a canonical covering map $G'_p \to G''_p$ sending $u\langle ab\rangle$ to $u\langle a, b\rangle$ for $u \in PSL_2(p)$. This is a two-fold covering and it is compatible with $\mathrm{Aut}(G'_p)$.

14. Some Subgroups of A'_5

We now describe the subgroups of A'_5 shown on Fig. 7. We have $E_5 = A'_4 \cap A''_4 = (A'_4)^+$. T is the normal subgroup of A'_5 which corresponds to Tutte's 8-cage. As is well known we have $E_5/T \cong A_6$, $(A'_5)^+/T \cong S_6$, and $A'_5/T \cong \mathrm{Aut}(S_6)$.

M is the normal subgroup of A'_5 which corresponds to the unique finite primitive 5′-object $(G(234), \mathrm{Aut}(SL_3(3))$. We have $A''_4/M \cong SL_3(3)$ and $A'_5/M \cong \mathrm{Aut}(SL_3(3))$.

For each prime $p \equiv \pm 1 \pmod{16}$ we denote by N'_p and N''_p two normal subgroups of A'_4 which correspond to the primitive 4′-object $(G_p, PSL_2(p))$.

By B' and B'' we denote two 1-regular subgroups of A'_4 which are not conjugate in A'_4. Each of B' and B'' has eight conjugates in A'_4 and of course B' and B'' are conjugate in A'_5 (Theorem 4). Let H' resp. H'' be the intersection of the conjugates of B' resp. B''.

Finally we put $K = H' \cap H''$.

Proposition 29. *Heawood's graph is a unique minimal* $(4', 1')$*-object. We have* $H' \leqslant (A'_4)^+$ *and so every* $(4', 1')$*-object is a covering of Heawood's graph. Further,* $(B' : H') = 42$, $(A'_4)^+/H' \cong PSL_2(7)$, *and* $A'_4/H' \cong PGL_2(7)$.

Proof. Let Π be the projective plane over the Galois field $GF(2)$. The vertices of Heawood's graph H are the points and lines of Π. A point-vertex is joined by an edge to a line-vertex if an only if this point and line are incident in Π. Thus H is a connected, bipartite, cubic graph of girth 6, having 14 vertices. H is 4-regular, $\mathrm{Aut}(H) \cong PGL_2(7)$ and $\mathrm{Aut}(H)$ has 1-regular subgroups. Thus H is a $(4', 1')$-object. Since $PSL_2(7)$ acts primitively on point-vertices as well as on line-vertices, it follows that H is minimal as a $(4', 1')$-object.

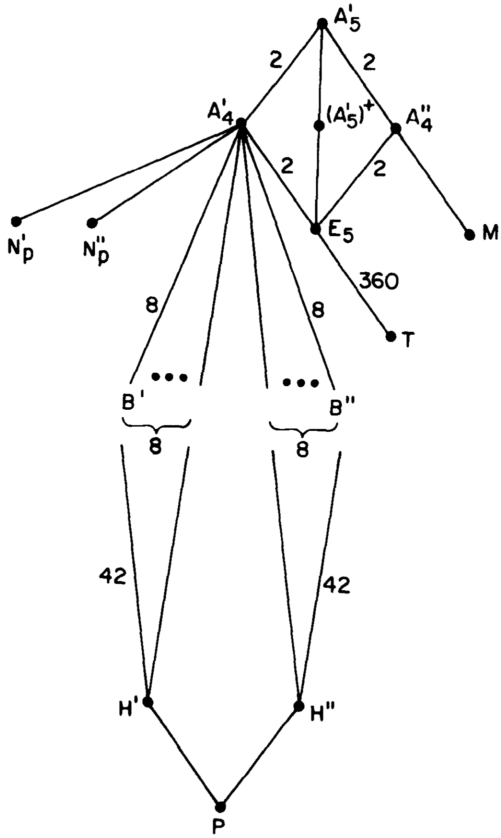

FIGURE 7

Now, let (G, A, B) be a $(4', 1')$-object, thus $B \leqslant A \leqslant \mathrm{Aut}(G)$, A is 4-regular, and B is 1-regular. Let $f\colon \Gamma_3 \to G$ be any covering. Then we can lift groups A and B to Γ_3. More precisely, we define $\tilde{A}$ to be the subgroup of $\mathrm{Aut}(\Gamma_3)$ consisting of all automorphisms ϕ of Γ_3 for which there exists an automorphism $\alpha \in A$ such that $\alpha \circ f = f \circ \phi$. One defines $\tilde{B}$ similarly. Then $\tilde{A}$ is 4-regular and $\tilde{B}$ is 1-regular (see [4]). Hence we may assume that $\tilde{A} = A'_4$ and $\tilde{B} = B'$, say. Let $g\colon \tilde{A} \to A$ be the homomorphism induced by the covering $f\colon \Gamma_3 \to G$. Then $\ker g = K \lhd A'_4$ and $K \leqslant B'$.

This implies that $K \leqslant H'$.

Hence the $(4', 1')$-object associated by Theorem 1 to the small subgroup H' of A'_4 is a unique minimal $(4', 1')$-object. Thus this object must be H. Since the Heawood's graph is bipartite we must have $H' \leqslant (A'_4)^+$. All other assertions follow from the properties of Heawood's graph.

PROPOSITION 30. *There is a unique minimal $(5', 1')$-object; this is the object (G_p, A_p) associated to the small subgroup $P = H' \cap H''$ of A_5'. We have $P \subset (A_5')^+$ and so every $(5', 1')$-object has the following properties*:

(1) *is a covering of Heawood's graph*;
(2) *is a covering of G_p*;
(3) *contains a $4'$-regular subgroup*;
(4) *is bipartite.*

Proof. We omit it because it is very similar to the proof of the previous proposition.

15. EMBEDDING A $1'$-OBJECT INTO AN ORIENTABLE SURFACE

Let (G, A) be a $1'$-object. Here we shall consider the graph G realized as a topological space in the usual way, see [10, pp. 1–20].

We know that there are precisely two A-conjugacy classes of A-shunts (Lemma 2); we fix one of these classes and refer to it and the shunts belonging to it as *positive.* If (a, α) is an A-shunt then its trajectory is either a circuit or a doubly infinite path in G. If $\{u, v\}$ is an edge belonging to this trajectory and $\alpha(u) = v$ then we assign to this edge the orientation from u to v. This oriented trajectory will be called an A-trajectory; it is *positive* if the corresponding shunt (a, α) is positive.

Now let $\{a, b\}$ be an edge of G. Let (a, α) and (b, β) be positive A-shunts such that $\alpha(a) = b$ and $\beta(b) = a$. Then Proposition 6 implies that $\beta(a) \neq \alpha^{-1}(a)$ and so (a, α) and (b, β) have different trajectories. Hence it follows that every edge of G belongs to precisely two positive A-trajectories and the orientations assigned to it in these A-trajectories are opposite. Thus if we glue a 2-cell along each positive A-trajectory then we shall obtain an orientable surface.

All positive A-trajectories have the same length, say, n; if these trajectories are doubly infinite paths then $n = \infty$. If (a, α) is a positive A-shunt then n is the order of α.

PROPOSITION 31. *Let (G, A) be a $1'$-object, (a, α) an A-shunt and n the order of α. If G is finite with v vertices then there is an embedding of G into an orientable closed surface S of genus*

$$g = 1 + \frac{n-6}{4n} v.$$

Moreover, the action of A on G extends to an action of A on S.

Proof. All the claims follow from the preceding discussion except the formula for the genus. Let e be the number of edges of G and f the number of 2-cells used in constructing S. Each of these 2-cells together with its boundary is an n-gon. Hence we have $nf = 3v = 2e$. The Euler characteristic of S is

$$\chi = v - e + f = \frac{6-n}{2n} v.$$

since $2g = 2 - \chi$ we are done.

For instance, (K_4, A_4) is a $1'$-object. The A_4-shunts have order 3 and we obtain $g = 0$. Thus we obtain the ordinary embedding of K_4 into the sphere.

Since g is an integer, this proposition can be used to show that certain graphs do not have 1-regular groups of automorphisms.

16. Some Open Questions

We first give an example of a 3-regular graph which has no 2-regular group of automorphisms.

Let

$$a = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 4 & -3 \\ 3 & -4 \end{pmatrix}$$

be in $PLG(13)$. Then in fact a, b, c are in $PSL_2(13)$ but y is not.

Since

$$ay = \begin{pmatrix} -2 & 2 \\ -1 & -3 \end{pmatrix}$$

has order 14 and

$$(ya)^2(yayb)^3 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

has order 13 it is clear that the elements a, b, c, y generate $PGL_2(13)$.

These elements satisfy

$$a^2 = b^2 = c^2 = y^2 = (ab)^2 = (bc)^2 = (cy)^2 = (ac)^3 = 1, \qquad yby = bc$$

Hence the amalgam $(\langle a, b, c\rangle, \langle b, c, y\rangle)$ is isomorphic to (X_3, Y_3). We define a graph G as follows: its vertices are the cosets $u\langle a, b, c\rangle$ for $u \in PGL_2(13)$ and its edges are $\{u\langle a, b, c\rangle, v\langle a, b, c\rangle\}$ for u, $v \in PGL_2(13)$, $u^{-1}v \in \langle a, b, c\rangle$

$y\langle a, b, c\rangle$. Then $PGL_2(13) = \text{Aut}(G)$ and G is a $3'$-object. Since $|PGL_2(13)| = 13 \cdot 168$ and $|\langle a, b, c\rangle| = 12$ it follows that G has $13 \cdot 14 = 182$ vertices. Since $PSL_2(13)$ is simple it is clear that it is the unique subgroup of $PGL_2(13)$ of index 2. The graph G is bipartite because $\langle a, b, c\rangle \leqslant PSL_2(13)$ and $\text{Aut}(G)$ is not generated by the vertex-fixers.
Consequently G has no 2-regular group of automorphisms.

We shall now list a few questions whose answers still evade us.

PROBLEM 1. Construct a finite graph G which is 5-regular and has no 4-regular groups of automorphisms.

PROBLEM 2. Is there a finite connected cubic graph G such that $\text{Aut}(G)$ is 2-regular of type $2''$?

PROBLEM 3. Is there a finite connected cubic graph G such that $\text{Aut}(G)$ is 4-regular of type $4''$?

PROBLEM 4. Is every minimal s'- or s''-object finite?

PROBLEM 5. Let A be an s-regular subgroup of $\text{Aut}(\Gamma_3)$ and B a t-regular subgroup of $\text{Aut}(\Gamma_3)$. Assume that $A \cap B$ is a 1-regular group. Is it true that the subgroup $\langle A, B\rangle$ of $\text{Aut}(\Gamma_3)$ is a free product with amalgamation of A and B with $A \cap B$ amalgamated?

PROBLEM 6. Let A and B be two subgroups of $\text{Aut}(\Gamma_3)$ of type $4'$ such that $A \cap B$ is 1-regular. Is it true that the subgroup $\langle A^+, B^+\rangle$ is simple?

PROBLEM 7. Determine the girth of the graphs G_p (see Section 10). (It is known [2] that the girth of G_{17} is 9.)

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